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UNIVERSITY OF JAMMU
JAMMU

SELF LEARNING MATERIAL

M. A. ECONOMICS

Title : **MATHEMATICAL METHODS IN ECONOMICS**

2020 Onwards

COURSE CODE : ECO-103	UNIT : I - IV
SEMESTER : I	LESSON NO. : 1-16

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MATHEMATICAL METHODS IN ECONOMICS

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DETAILED SYLLABUS

C.No. ECO-103 **Title : Mathematical Methods in Economics**

Credits : 6 **Maximum Marks : 100**

(a) Semester Examination : 80

Duration of Examination : 3.00 hrs. **(b) Sessional Assessment : 20**

MATHEMATICAL METHODS IN ECONOMICS

Syllabus for the Examination to be held in December 2019 to December 2021

Preamble- The aim of this course is to train students in the use of mathematical tools to understand concepts in economics presented in the form of mathematical models and express economic ideas in the same form. The course is intended to enable the students to utilize these tools in subsequent courses in the II, III and IV semesters especially those courses where the use of mathematics has become a norm.

UNIT-I EQUATIONS AND DIFFERENTIATION

Numbers-natural, Integers, rational, irrational, complex, linear equations. Mathematical operations with Matrices, solution of simultaneous equations: Rank of the matrix, matrix inversion. Quadratic Equations, Eigen roots and Eigen Vectors. Concepts of limit and continuity, Economic examples and applications. Principles of differentiation, rules of differentiation, differentiation of implicit function, parametric function.

UNIT-II CALCULUS

Partial and total differentiation, Expansion by Taylor Series. Allied economic applications. Maxima and Minima- constrained and unconstrained, economic application. Principles of integration: Indefinite and definite. Application in economics theory Economic application- Derivation of Consumers Surplus, Producers Surplus, Profit and utility maximization with one good, product and input, Derivation of demand curves for inputs, goods etc., Comparative statics and allied economic applications: combining calculus-Derivation of Slutsky Equation, IS-IM model, Cobb Douglas and CES production functions, elasticity of demand, supply substitution.

UNIT-III DIFFERENCE AND DIFFERENTIALS

Differential Equations: definitions and concepts; Solution of first order and second order differential equations, Difference equations: definitions and concepts; Solution of first order and second order difference equations, Simultaneous Differential equations ad phase diagrams, Application of difference and differential equations in Economics-Cobweb model, foreign trade multiplier model, Market model with stocks- National Income Model.

UNIT-IV LINEAR PROGRAMMING, OPERATIONS AND APPLICATIONS

Linear programming- Basic concepts; functions of a LP problem; Nature of feasible, basic and optimal solutions; Solution of a LP problem through graphical and simplex methods (Slack, Surplus and artificial variables); Formulation of Dual and its interpretation; Input-Output Analysis: Introduction, Input-Output transaction table, the technological Co- efficient matrix, solution of open model, The Hawkins-Simon Conditions, solution for 2 and 3 industries, determination of equilibrium prices.

NOTE FOR PAPER SETTER :

There shall be two types of questions in each Unit - four short answer type (each of 250 words) and two medium answer type (each of 500 words). The candidate will have to attempt two short answer type questions and one medium answer type question from each Unit. Each short answer type question shall carry 4 marks and each medium answer type question carry 12 marks.

BASIC READING LIST:

1. Allen, R.G.D. (1976). Mathematical Analysis for Economists, Macmillan.

2. Chiang, A.C.(1974). Fundamental Methods of Mathematical Economics, McGraw Hill and Kogakusha, New Delhi.

3. Mehta & Madnani (1992). Mathematical for Economists, S. Chand, New Delhi.

4. Samuelson, P.A. (1967). Foundations of Economic Analysis, McGraw Hill, Tokyo.

5. Henderson & Quandt, Microeconomics: A Mathematical Approach, Tata McGraw Hill.

6. Baumol, W.J.(l970). Economic Dynamics, Macmillan, London.

7. Leonard and Von Long (1978). Introduction to Maths for students of Economics, Cambridge.

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1.1 INTRODUCTION

The concept of numbers and the structure of numbers that make up the number system are basic to the use of calculus - the most important branch of Mathematics useful for practical purpose especially for the students of economics. The part of number system that is most applicable and useful in economics is the system of real numbers; we will examine their types and operations.

1.2 OBJECTIVES

After reading this unit you should be able to :-

* The number system
* Solve the linear equations
* How they are used in economics

1.3 THE NUMBER SYSTEM

1.3.1 Natural numbers and whole numbers

Numbers of the form 1, 2, 3, 4 which are frequently used in counting are called natural numbers [or whole numbers if 0 is also included in it] and the set of all natural numbers is denoted by the symbol N or Z. Natural numbers are also called cardinal numbers or positive integers.

N (or z) = {1, 2, 3 --} = {x : x is a natural number}

= {x: x $\in$ z}

1.3.2 Integers

Integers are the numbers consisting of all natural numbers, zero and negatives of natural numbers. The set of integers is denoted by I. Thus,

I = {....-3, -2, -1, 0, 1, 2, 3, ...}

I = {x : x $\in$ 1}

1, 2, 3 .. are called positive integers (+ve) and -3, -2, -1 are called negative integers (-ve). The set of all positive integers is denoted by I^+, the set of all negative integers by I^- and the set of all integers excluding 0 {i.e. the set of all non-zero integers} is denoted by I. Thus

I^+ = {1, 2, 3, ..} = {x : x is a +ve integer}

= {x, x $\in$ I^+}

I^- = {..-3, -2, -1, ...} = {x : x is -ve integer}

= {x, x $\in$ I^-}

I_0 = {....-3, -2, -1, 1, 2, 3, ..}

= {x : x is an integer excluding 0}

= {x, x $\in$ I_0}

1.3.3 Rational numbers

All numbers of the form a/b, where a and b are integers and b $\neq$ 0 are called rational numbers. These can be expressed either as finite decimals or recurring (infinite) decimals. For example 1/5 = 0.2, 1/2 = 0.5 and 1/3 = .333 .... The set of all rational numbers is denoted by Q. Thus,

Q = {a/b: a and b are integers and b $\neq$ 0}

Q^+ = {All positive rational numbers}

= {x: x is a +ve rational number}

Q^{-1} = {All negative rational numbers}

= {x: x is a -ve rational number}

Q_0 = {All rational numbers excluding 0}

= {All non-zero rational numbers}

= {x : x is a rational number + $x \neq 0$}

1.3.4 Irrational numbers

Numbers which are not rational numbers or which cannot be expressed as the ratio of two integers are called irrational numbers. For example, $\pi = \frac{22}{7} = 3.14159$, $\sqrt{2}$, $\sqrt{3}$ - are irrational numbers.

1.3.5 Real numbers

The totality of all rational plus irrational numbers is called the set of all real numbers and is denoted by R. Thus,

R = {All rational numbers + All irrational number}

= {x : x is a real number} = {x: $x \in R$}

R^+ = {All positive real numbers}

= {x : x is a +ve real number}

= {x : $x \in R^+$}

R^{-1} = {All negative real numbers}

= {x : x is a -ve real number}

= {x : $x \in R^-$}

R_0 = {All non-zero real numbers}

= {x : x is a real number and $x \neq 0$}

= {x : $x \in R_0$}

1.3.6 Imaginary numbers

Numbers of the form $\sqrt{-a}$, where is a positive integer are called imaginary numbers. Thus, square roots of negative numbers are called imaginary numbers. For example $\sqrt{-2}$, $\sqrt{-3}$ ... are called imaginary numbers.

1.3.7 Complex numbers

Numbers of the form a + ib where a and b are real numbers and i = $\sqrt{-1}$ are called complex numbers. The set of all complex numbers may be denoted by C. Thus,

C = {a/b : a and b are real numbers}

= {All real numbers + All imaginary numbers + All combinations of these two}

For example, 5 + 2i, 5 - 2i, 7 + $\sqrt{-5}$ ... are all complex numbers

1.4 LINEAR EQUATIONS

In many cases, the relationship between economic variables may be linear. A demand schedule for a good may reveal the linear relationship between the amount demanded of the good and its price. Similarly, aggregate consumption in a country may be linearly related to its aggregate disposable income. Moreover, in econometric models, linear regression equations are widely employed.

1.4.1 Definition of Equation

A polynomial of nth degree in x equated to zero is termed as an equation of nth degree in x. The unknown quantity in the equation is called variable. Thus, ax + b =0 ($a \neq 0$) is an equation of first degree in x. Equations of first degree in x are also called the linear equations. Similarly, $ax^3 + bx^2 + cx + d = 0$ ($a \neq 0$) is an equation of 3[rd] degree in x or a cubic equation.

1.4.2 The Root of an equation

A value of the variable, which renders both sides of the equation identical is called a root of the equation. For example 3 is the root if x + 3 = 6; 3 & 5 are the roots of $x^2 - 8x + 15 = 0$.

1.4.3 The Degree of an equation

The degree of an equation is the index of the highest power unknown quantity or the variable involved of the equation when the equation has been cleared of all the radicals and the denominator involving the variable.

For example

$$\sqrt{x} + \frac{1}{\sqrt{x}} = 4$$

Multiplying by $\sqrt{x}$, we have

$$\left(\sqrt{x}\right)\left(\sqrt{x}\right) + \frac{1}{\sqrt{x}}\left(\sqrt{x}\right) = 4\sqrt{x}$$

$$x + 1 = 4\sqrt{x}$$

Squaring both sides

$$(x+1)^2 = 16\ \left(\sqrt{x}\right)^2$$

$$x^2 + 2x + 1 = 16x$$

$$x^2 - 14x + 1 = 0$$

Clearly, the highest power of x is 2

$\therefore$ The equation is of degree 2 (quadratic)

Example : $\dfrac{7x+8}{3x+1} = \dfrac{15}{4}$

Solution : By cross multiplying, we get

$$4(7x+8) = 15(3x+1)$$

$$28x+32 = 45x+15$$

$$28x - 45x = 15-32$$

$$-17x = -17 \qquad \therefore \; x = 1$$

Example : $\dfrac{1}{x+1} + \dfrac{3}{x+4} = \dfrac{4}{x+3}$

Solution : We split the term on the right hand side i.e.

$$\frac{4}{x+3} = \frac{1}{x+3} + \frac{3}{x+3}$$

The given equation can be written as

$$\frac{1}{x+1} + \frac{3}{x+4} = \frac{1}{x+3} + \frac{3}{x+3}$$

or $\dfrac{1}{x+1} - \dfrac{3}{x+3} = \dfrac{3}{x+3} - \dfrac{3}{x+4}$

or $\dfrac{x+3-x-1}{(x+1)(x+3)} = \dfrac{3x+12-3x-9}{(x+3)(x+4)}$

or $\dfrac{2}{(x+1)(x+3)} = \dfrac{3}{(x+3)(x+4)}$

or $\dfrac{2}{x+1} = \dfrac{3}{x+4}$

or $2x + 8 = 3x + 3$

$2x - 3x = 3-8$

$-x = -5$

$x = 5$

Self-Assessment - I

1. What do you understand by the term Integers ?

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2. Why the set of Integers is known as a Universal Set ?

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$\therefore$

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3. What are imaginery numbers and complex numbers ?

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1.4.4 Simultaneous linear equations in two variables

consider the equations $a_1x + b_1y+c_1 = 0$

$$a_2x + b_2y+c_2 = 0$$

There is a particular pair of values of x and y which solves both equations, then the equations are called simultaneous equations. The pair of values is called the solution of two equations. There are various methods of solving the simultaneous equations.

Method 1: Equalization of co-efficients method

In this method, we multiply the two equations by suitable number that the co-efficients of either of the two variables become equal in the equations. Then, by adding the two equations or by subtracting, we get on equation in one variable.

Method 2 : Method of comparison

In this method we find the value of one variable in terms of the other from both the equations and equate them. Thus, we get an solution in one variable only.

Method 3 : Method of substitution

We find the value of one variable in terms of other from one equation and substitute that value in the other, we get an equation in one variable.

Example : Solve the equations

$$2x + y = 7$$

$$x + 3y = 6$$

Solution 1 : The given equations are

Method 1 $2x + y = 7$ - (i)

$x + 3y = 6$ - (ii)

Multiply equation (ii) by 2, we get

$2x+6y = 12$ - (iii)

Subtracting (iii) from (i), we get

$$5\ y = 5$$

$$y = 1$$

Substituting this value of y in equation (i)

$$2x + 1 = 7$$

$$2x = 6$$

$\therefore$ Required solution is $x = 3, y = 1$

Method 2 : The given equations are

$2x + y = 7$ - (i)

$x + 3y = 6$ - (ii)

from (i), $2x = 7-y$

$x = \frac{7-y}{2}$

from (ii), $x = 6-3y$

$$\therefore \frac{7-y}{2} = 6-3y$$

or $7-y = 2(6-3y)$

or $7-y = 12-6y$

$5y = 12-7$

$$y = \frac{5}{5} = 1$$

from e.g. (i), we have

$2x + y = 7$

$2x + 1 = 7$

$2x = 7-1$

$x = 3$

Required solution is $x = 3, y = 1$

Method 3: The given equations are

$2x + y = 7$ - (i)

$x + 3y = 6$ - (ii)

From (ii), we have $x = 6 - 3y$

Substituting in (i), we get

$2(6 - 3y) + y = 7$

$12 - 6y + y = 7$

$-5y = 7-12$

$-5y = -5$

$y = 1$

If $y = 1$, $x = 6-3y = 6-3 = 3$

Required Solution is $x = 3, y = 1$

Rules of cross multiplication

Consider the equations

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

Note : Right hand side of these equations must be zero, when we apply this rule

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{a_2 c_1 - a_1 c_2} = \frac{1}{a_1 b_2 - b_1 a_2}$$

$$\therefore \quad x = \frac{b_1 c_2 - c_1 b_2}{a_1 b_2 - b_1 a_2}$$

$$y = \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - b_1 a_2}$$

Example : Solve the equations

$4x - 5y = -8$

$2x - 3y = -6$

Solution

$$4x - 5y + 8 = 0$$

$$2x - 3y + 6 = 0$$

$$\frac{x}{-30+24} = \frac{y}{16-24} = \frac{1}{-12+10}$$

$$x = \frac{-6}{-2} = 3$$

$$y = \frac{-8}{-2} = 4$$

$\therefore$ x = 3, y = 4

1.4.5 Simultaneous linear equations in three variables.

The method of solving linear equations in three variables is similar to that used for solving two simultaneous equation. First, we eliminate one of the three variables from the given equations and arrive at linear equations in two variables. By solving these two equations we get the values of the variables. Then, by substituting the values of these two variables in any of the three equations and get the value of the third remaining variables.

Example : Solve the equations

$$x + y + z = 10$$

$$2x + 3y + 4z = 33$$

$$3x - y + z = 8$$

Solutions : The given equations are

$$x + y + z = 10 \qquad \text{- (i)}$$

$$2x + 3y + 4z = 33 \qquad \text{- (ii)}$$

$$3x - y + z = 8 \qquad - (iii)$$

Multiply (i) by 4, we get

Subtracting (ii) from (iv), we get

$$2x + y = 7 \qquad - (v)$$

Again, multiply (iii) by 4, we get

$$12x - 4y + 4z = 32 \qquad - (vi)$$

Subtracting (vi) from (ii), we get

$$-10x + 7y = 1 \qquad - (vii)$$

Multiplying (v) by 5

$$10x + 5y = 35 \qquad - (viii)$$

Adding (vii) & (viii)

$$12y = 36$$

$$y = 3$$

From (viii), $10x = 5 \quad 3 = 35$

$$10x = 35 - 15$$

$$x = \frac{20}{10} = 2$$

From (i), $2+3+z = 10$

$$Z = 10-5 = 5$$

$\therefore$ $x = 2$, $y = 3$, $z = 5$ is the required solutions

Self-Assessment-II

Solve the following :

a) $\frac{x}{x-1} - \frac{x-1}{x-2} = \frac{x-2}{x-3} - \frac{x-3}{x-4}$

b) $3x + 2y = 9$; $x + 3y = 10$

1.5 ECONOMIC PROBLEMS INVOLVING LINEAR EQUATIONS

1.5.1 Demand Condition

It involves the problems related to the demand conditions in an economy.

Example : Henry Schultz estimates the demand curve for sugar in the U.S. during the period 1915-1929 to be D = 135 - 8p where D stands for quantity demanded and p stands for price

a) Find the price if the quantity demanded is 93

b) How much sugar would be demanded if it were a free good.

c) Find the amount demanded if price is 7

d) What is the highest price any one will pay.

Solution : The estimated demand function is D = 135 - 8p - (i)

a) Here, we are given is D = 93 substituting this value of D in (i), we have

$$93 = 135 - 8p$$

$$8p = 135 - 93 = 42$$

$$p = \frac{42}{8} = 5.25$$

b) If sugar were a free good, p = 0

$$D = 135 - 8p$$

$$D = 135$$

c) If p = 7

$$D = 135 - 8\,(7)$$

$= 135 - 56 = 79$

d) Putting D = 0, in the equation D = 135-8p

$D = 135 - 8p$

$8p = 135$

$$p = \frac{135}{8} = 16.875$$

Thus, if the price is 16.875 the amount demanded is zero. Therefore, it is clear that the price must be something less than 16.875 if any amount of sugar is to be sold.

1.5.2 Equilibrium Condition

It involves the problems related to the market forces and their intervention in the economy. It dealt with the equilibrium in the real world with inclusion and exclusion of taxes.

Example: If the demand and supply equations are given by p = 12-3q and p $= \frac{3}{2}q + 2$. Find the equilibrium price and quantity before and after tax t = $\frac{1}{2}$ per unit imposed.

Solution: (i) The demand and supply equations are given by p = 12-3q

and $p = \frac{3}{2}q + 2$ - ii)

For equilibrium

$D = S$

$$\frac{3}{2}q + 2 = 12 - 3q$$

$$\frac{3}{2}q + 3q = 12 - 2$$

$$\frac{9}{2}q = 10$$

$$q = \frac{20}{9}$$

Put in (i)

$$p = 12 - 3\left(\frac{20}{9}\right)$$

$$= 12 - \frac{20}{3} = 16/3$$

$\therefore$ Before tax, equilibrium price = 16/3 and equilibrium quantity = 20/9

ii) When tax t = 1/2 per unit is imposed then the new demand and supply equations are

$$p_1 = 12 - 3q_1 \qquad \text{- (iii)}$$

$$p_1 - \frac{1}{2} = \frac{3}{2}q_1 + 2 \qquad \text{- (iv)}$$

$$12 - 3q_1 - \frac{1}{2} = \frac{3}{2}q_1 + 2$$

$$\frac{3}{2}q_1 + 3q_1 = 12 - 2 - 1/2$$

$$\frac{9}{2}q_1 = \frac{19}{2}, \therefore q_1 = \frac{19}{9}$$

$$\therefore p_1 = 12 - 3\left(\frac{19}{9}\right)$$

$$=12-\left(\frac{19}{3}\right)$$

$p_1 = 17/3$

$q_1 = 19/9$ equilibrium price and quantity

1.5.3 Consumption Function

This topic includes income, consumption, savings and their respective functions with the addition of their propensities to save and consume.

Example : Given the consumption function C = Rs (40 + 621 Y) Crore

Where C is consumption and Y is income

i) Find the level of consumption when Y = Rs. 325 crore. Here find the average propensity to consume when income is 325 cr.

ii) Find the level of Y if consumption is Rs. 250 crore.

iii) Assuming Y = C + S where S is saving find S when Y= Rs. 250 cr. Hence find APC and APS and show that APC + APS = 1

Solution : i) C = 40 + .621 Y

When Y = Rs. 325

C = 40 + (.621) (325)

= 40 + 201.825

= 241.83 cr

Now APC when income is Rs. 325 cr.

$$= \frac{C}{Y} = \frac{241.83}{325} = .744$$

ii) When C = Rs. 250 cr.

$250 = 40 + .621Y$

$.621y = 210$

$$Y = \frac{210}{.621} = Rs.338.16cr$$

$Y = C + S$

$S = Y - C$

$= Rs.\ (250-195.25)\ cr.$

$= Rs.\ 55.25\ cr$

$$APC = \frac{C}{Y} = \frac{195.25}{250} = 0.78$$

$$APS = \frac{S}{Y} = \frac{55.25}{250} = 0.22$$

$\therefore\ APC + APS = .78 + 0.22$

$= 1$

1.5.4 Multiplier Models

It includes multipliers Models of tax, expenditure and balanced budget by considering the equilibrium level of income.

Example :

a) Derive the equilibrium level of income from the following model

$C = C_o + bY_d,\ Y_d = Y - T_x$

$T_x = T_{xo} + tY \quad I = I_o + aY,\ G = G_o$

b) What are the expenditure, tax and balanced budget multiplier for this model

Solution : We know that equilibrium level of income occurs where

$Y = C + I + G$

$= C_o + bYd + I_o + aY + G_o$

$= C_o + b\,(Y\text{-}T_{xo} + tY) + I_o + aY + G_o$

$Y = C_o + bY - bT_{xo} - btY + I_o + aY + G_o$

$Y - bY - aY + btY = C_o - bT_{xo} + I_o + G_o$

$(1 - b - a + bt)\,Y = C_o - bT_{xo} + I_o + G_o$

$$Y = \frac{C_o - bT_{xo} + I_o + G_o}{1\text{-}b\text{-}a + bt}$$

b) Expenditure multiplier measures the change in income from an autonomous change in spending

$$\therefore = \frac{\Delta Y}{\Delta I} = K_a = \frac{1}{1 - b - a + bt}$$

The tax multiplier measures the change in income from equal autonomous change in T_{xo}.

$$\frac{\Delta Y}{\Delta T_x} = \frac{K}{t_x} = \frac{-b}{1 - b - a + bt}$$

Balanced budget multiplier measures changes in income from equal autonomous change in government spending and taxes

$K_a + K_t, = K_b$

$$\therefore\ K_b = \frac{1\text{-}b}{1 - b - a + bt}$$

Self-Assessment - III

1. The supply curve of a commodity is given to be S = ap - b.

a) Find the ammount supplied if $p = \frac{3b}{a}$

b) What will be price if the amount supplied is 7a - 3b.

c) What is the lowest price at which the commodity will be sopplied.

IS - LM Analysis

Heavily dependent on introduction to mathematical economics - Edward T. Dowling

The IS schedule is a locus of points representing all the different combinations of interest rates and income levels consistent with equilibrium in the goods (commodity) market. The LM schedule is a locus of points representing all the different combinations of interest rates and income levels consistent with equilibrium in the money market. IS-LM analysis seeks to find the level of income and the rate of interest at which both the commodity market and the money market will be equilibrium. This can be accomplished with the techniques used for solving simultaneous equation. IS-LM analysis deals explicitly with the interest rates and incorporates its effect into the model.

IS and LM schedules

Taking a closed economic system with no external sector and no government activity gives us an identity.

$$Y \equiv C + I$$

Where $\equiv$ is the identically equals sign, Y is national income, C is consumption and I is investment.

$$Y = C + I \text{ thus } Y-C = I$$

but $\quad Y-C = S$

where S is saving, so

$$S = I$$

The Keynesian model assumes that S is a function of Y, and I is a function of the rate of interest r, i.e.

$$S = sY$$

where s is the marginal propensity to save, MPS and

$$I = I^{*} - \alpha r$$

where I^{*} is a constant and   is the parameter of the investment function, note that $\alpha > 0$ so I decreases as r increases.

The IS schedule is a function along which I = S or

$$I^{*} - \alpha r = I^{*} \qquad - (1)$$

Hence the IS schedule is a linear function in Y and r. Along this schedule, the goods market is in equilibrium. To determine equilibrium Y and r for this economic system, another function in Y and r is required. This is obtained from the money market.

The money market is in equilibrium when the supply of money M_S is equal to demand for money M_D. In Keynesian terms M_D depends upon the transactions, precautionary and speculative motives for holding cash.

$$M_D = M_{D1} + M_{D2}$$

Where M_{D1} = the transactions and precautionary demand for money and M_{D2} = the speculative demand for money. Keynes assumes that M_{D1} is an increasing function of Y and M_{D2} varies inversely with r. If $M_{D1} = \beta Y$ and $M_{D2} = k\text{-}gr$ where β, k and I are positive constant and M_S is constant at M^{*}, then the equilibrium in the money market will occur when

$$M_s = M_{D1} + M_{D2}$$

$$M^{*} = \beta Y + k - gr$$

$$\beta Y - gr = M^{*} - k \qquad - \quad (2)$$

This function gives all possible combinations of Y and r which bring equilibrium to the money market and is called the LM schedule.

Equilibrium will occur in the money and goods market when equation 1 and 2 hold i.e. when

$$sY + \alpha r = I^*$$

and

$$\beta Y - gr = M^* - k$$

We now have two equations in two unknown i.e. Y and r, whose solution, which can be found in the usual way, will give the equilibrium Y and r for this simple economic system.

Example : Give in the following information about a closed economy

Consumption C = 100+0.8Y

Investment I = 1200 - 30r

Where r is the rate of interest

Precautionary and transaction demand for money

$M_{D1} = 0.25Y$

Speculative demand for money

$M_{D2} = 1375 - 25r$

Money supply $M_S = 2500$

Find the equilibrium values of Y and r

Solution : $Y = C + I$

$$C = 100 + 0.84$$

and

$$I = 1200 - 3r$$

$$Y = 100 + 0.84 + 1200 - 3r$$

$$(1-0.8)\ Y = 1300 - 3r$$

$0.2Y \quad = 1300 - 3r$

$$= \frac{1300}{.2} - \frac{3}{.2}r$$

$= 6500 - 150r$ - (1)

The money market is in equilibrium when $M_S = M_D$ but $M_D = M_{D1} + M_{D2}$

thus $M_S = M_{D1} + M_{D2}$ in equilibrium so

$2500 = 0.25Y + 1375 - 25r$

$0.25Y = 1125 + 25$

$$Y = \frac{1125}{.25} + \frac{25}{.25}r$$

$= 4500 + 100r$ - (2)

Thus from (1) and (2), we get

$6500 - 150r = 4500 + 100r$

$6500 - 4500 = 100r + 150r$

$2000 = 250r$

$r = 8$

When $r = 8$, $Y = 4500 + 100(8)$

$= 4500 + 800$

$= 5300$

1.6 SUMMARY

In this lesson, we have

1) Explained the number system

Numbers	In symbols
Natural	N = {x : x $\in$ z}
Integers	I = {x : x $\in$ I}
Rational	Q = {a/b : a & b are integers but b $\neq$ 0}
Irrational	= 22/7
Real	R = { x : x $\in$ R_o}
Complex number	a + ib

2) Discussed the Linear equations

3) Used the linear equations for solving the economic applications

1.7 LESSON END EXERCISE

Solve the following

i) 4x + 3y b= 7 ; 3x + 2y = 9

ii) $\frac{3}{x}+\frac{2}{y}=1\ ;\ \frac{1}{x}+\frac{1}{y}=10$

iii) $\frac{x-4}{3}=\frac{y-1}{4}\ ;\ \frac{4x-5y}{7}=x-7$

iv) x - y - z = 1, y- z - x = 1, z - x - y = 1

v) x + 2y + 3z = 1, 3x+2y+4z = 2, 3x+4y + 3z =3

vi) The demand curve for a commodity is given as D = 20-5p and supply curve is S = 6p - 21. Find the equilibrium price and quantity

vii) If the demand and supply equations are given by p =12-3q and p = . Find the equilibrium price and quantity before and after tax t = 1/2 is imposed

1.8 SUGGESTED READINGS

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MATHEMATICAL OPERATIONS WITH MATRICES, SOLUTION SIMULTANEOUS EQUATION, RANK OF THE MATRIX, MATRIX INVERSION

STRUCTURE

2.1. INTRODUCTION

The British mathematician Arthur Cayley was the first person to formulate general theory of matrices in 1857. He developed the properties of matrices as a pure algebraic structure. Later on, the world realized the importance of its applications in all important fields. Today, it is regarded as one of the most powerful and convenient technique in economics especially in input-output analysis, game theory, linear programming etc.

2.2 OBJECTIVES

After studying this lesson, you should be able to:-

* Type of Matrices
* Basic operations on Matrices
* Solve simultaneous equations
* Find the rank of matrices and matrix inversion
* Use of matrices in economics

2.3 MATRICES

2.3.1 DEFINITION OF MATRICES

A matrix is a rectangular array of numbers, parameters or variables, each of which has a carefully ordered place within the matrix. The numbers (parameters or variables) are referred to as elements of the matrix. The numbers in the horizontal line are called rows; the numbers in a vertical line are called the columns. The number of rows (r) and column (c) defines the dimension of the matrix (r x c) which is read "r by c". The rows number always precedes the column number.

A matrix of orders r x c can be shown as

$$A = \begin{bmatrix} a_{11} & a_{12} \dots a_{1c} \\ a_{21} & a_{22} \dots a_{2c} \\ \dots & \\ a_{r1} & a_{r2} \dots a_{rc} \end{bmatrix}$$

Each of the numbers a_{11} or a_{2c} or a_{ij} is known as the elements of A. a_{ij} is an element at the intersection of the i^{th} row and jth column. Elements a_{11}, a_{12}... a_{1c} are the elements of the first row and constitute row vector and similarly, a_{11}, a_{21}.... a_{r1} are the elements of the first column and constitute a column vector. As such a matrix can also be defined as an arrangement of row vector or column vector elements such as a_{11}, a_{22}.... a_{rc} are the elements in the principal diagonal of matrix A

$$Thus, A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} and\ B = \begin{matrix} b_{11} & b_{12} & b_{13} \\ \\ b_{21} & b_{22} & b_{23} \end{matrix}$$

are matrices of order 2x2 and 2x3 respectively

2.3.1 Type of Matrices

i) Square and rectangular matrix. A matrix in which the number of rows is equal to number of columns is called a square matrix. A matrix which is not a square matrix is called a rectangular matrix.

$$\text{e.g. if } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{are}$$

Square and rectangular matrices of order 2x2 and 2x3 respectively.

ii) Diagonal Matrix: A square matrix is said to be a diagonal matrix if its non-diagonal elements are zero

$$Example: A_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

are all examples of diagonal matrices of order 4,3 and 2 respectively.

iii) Scalar matrix: A diagonal matrix in which all the diagonal elements are the same is said to be scalar matrix.

$$Example: A_1 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 & 0 \\ r & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are all examples of scalar matrices of order 4,3 & 2 respectively.

iv) Unit Matrix : A scalar (or square) matrix in which all the diagonal elements are equal to one and is denoted by I

$$Example: I_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are all elements of identity matrices of order 4,3 & 2 respectively.

v) Null matrix: A matrix in which all its elements are zero is called a null matrix or a zero matrix and is denoted by 0 or 0_{mxn} etc.

$$Example: 0_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, 0_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$0_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0_5 = \begin{bmatrix} 0, & 0 \end{bmatrix}$$

are all examples of null matrix.

vi) Column matrix: A matrix which contains only a single column is called column matrix $e.g \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ *is a column matrix of order 2x1*

vii) Row Matrix: A matrix which contains only a single row is called a row matrix

e.g [3,5, 7] is a row matrix of order 1x3

viii) Transpose of a Matrix: A matrix obtained from the given matrix A, by interchanging its rows and columns and is called the transpose of the given matrix A and is denoted by A' or, A. Since A has m rows and n column, A' will have n rows and m columns.

$$Example: A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Then

$$A_1 = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

are examples of transposed matrices.

ix) Symmetric Matrix : A square matrix is said to be a symmetric matrix if it remains unchanged by the interchange of its rows and columns, i.e. $a_{ij} = a_{ji}$. In other words, in a symmetrical matrix A=A.

$$Examples: A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}_{3\times3}, \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 0 & 4 \\ -3 & 4 & 6 \end{bmatrix}_{3\times3}$$

$$A_3 = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

are examples of symmetric matrix. In all these matrices A= A

x) Skew -symmetric matrix : A square matrix A is said to be a skew-symmetric matrix if it changes only in sign, by inter-changing of its rows and columns i.e., (i, j) the elements of A= -(j, i) the elements of A, or $a_{ij} = -a_{ji}$

$$Examples: A_1 = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ g & -h & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -10 \\ +10 & 0 \end{bmatrix}$$

are examples of skew symmetrical matrices it may be noted that A_1 & A_2 will also be skew symmetrical matrices.

xi) Equality of matrices: Two matrices are said to be equal if and only if they are of the same order and the elements in the corresponding places of the two matrices are equal.

e.g. A $= \begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2^2 \\ 0 & -1 \end{bmatrix}$ are equal matrices

Self Assessment - I

1. What do you understand by Matrix. Differentiate between matrix and matrices.

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2. Define Unit Matrix and Null Matrix with the help of an example.

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2.4 BASIC OPERATIONS ON MATRICES

The basic operations in the matrix theory are

2.4.1 Scalar multiplication of matrix

2.4.2 Addition of matrices

2.4.3 Subtraction of matrices

2.4.4 Multiplication of matrices

2.4.1 Scalar multiplication of a matrix

Given a matrix A (square or rectangular) and a scalar (or constant) $k \neq 0$, KA is defined as the Scalar multiple of the matrix A and is obtained by multiplying every element of A by the Scalar K.

$$Illustration:(i)\,if\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{21} & a_{32} & a_{23} \end{bmatrix} then$$

$$KA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{33} & ka_{32} & ka_{33} \end{bmatrix} (k \neq 0)$$

2.4.2 Addition (or sum) of matrices

Let A= (a_{ij}) and B = (b_{ij}) be any two matrices. Then the addition A+B of the two matrices A+B is well defined if the two matrices A and B are of the same order, say m x n. If this condition is satisfied then value of addition is to add the corresponding elements of A&B

$$(i)\,if\ A = \begin{bmatrix} 1\ 2\ 3\ 4 \\ 6\ 7\ 8\ 2 \end{bmatrix}_{2x4} \quad B = \begin{bmatrix} 6\ 7\ 7\ 1 \\ 4\ 3\ 2\ 1 \end{bmatrix}_{2x4}$$

$$then\ A + B = \begin{bmatrix} 1+6 & 2+7 & 3+7 & 4+1 \\ 6+4 & 7+3 & 8+2 & 2+1 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 10 & 5 \\ 10 & 10 & 11 & 3 \end{bmatrix}_{2x4}$$

$$ii)\,2A + 3B = 2\begin{bmatrix} 1\ 2\ 3\ 4 \\ 6\ 7\ 8\ 2 \end{bmatrix} + 3\begin{bmatrix} 6\ 7\ 7\ 1 \\ 4\ 3\ 2\ 1 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 4 & 6 & 8 \\ 12 & 14 & 16 & 4 \end{bmatrix}+\begin{bmatrix} 18 & 21 & 21 & 3 \\ 12 & 9 & 6 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 2+18 & 4+21 & 6+21 & 8+3 \\ 12+12 & 14+9 & 16+6 & 4+3 \end{bmatrix}$$

$$=\begin{bmatrix} 20 & 25 & 27 & 11 \\ 24 & 23 & 22 & 7 \end{bmatrix}_{2x4}$$

2.4.3 Subtraction (or difference) of matrices

Let A = (a_{ij}) and B = (b_{ij}) mxn be the two matrices of the same mxn order i.e mxn, then the difference A-B of the two matrices A and B is defined as a matrix D= $[d_{ij}]$ mxn of the same order such that

D = A-B = $[a_{ij} - b_{ij}]_{mxn}$

$$Example\ :\ if\ A=\begin{bmatrix} 1 & -2 & 3 \\ 3 & -4 & 0 \\ -2 & 0 & -5 \end{bmatrix}_{3x3}\quad B=\begin{bmatrix} -2 & -3 & 4 \\ 4 & -5 & -3 \\ 2 & -3 & -4 \end{bmatrix}_{3x3}$$

are the two given matrices of the same order i.e. 3x3 then find (i) A-B (ii) 2A-5 B

Solution (i) $$\begin{matrix} A-B \\ [A+(-B)] \end{matrix}=\begin{bmatrix} 1 & -2 & 3 \\ 3 & -4 & 0 \\ -2 & 0 & -5 \end{bmatrix}+\begin{bmatrix} +2 & +3 & -4 \\ -4 & 5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 1+2 & -2+3 & 3-4 \\ 3-4 & -4+5 & 0+3 \\ -2-2 & 0+3 & -5+4 \end{bmatrix}=\begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 3 \\ -4 & 3 & -1 \end{bmatrix}_{3x3}$$

ii) $$2A+(-5)B=2\begin{bmatrix} 1 & -2 & 3 \\ 3 & -4 & 0 \\ -2 & 0 & -5 \end{bmatrix}+(-5)\begin{bmatrix} 2 & 3 & -4 \\ -4 & 5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & -6 \\ 6 & -8 & 10 \\ -21 & 0 & -10 \end{bmatrix} - \begin{bmatrix} -10 & -15 & 20 \\ 20 & -25 & -15 \\ 10 & -15 & -20 \end{bmatrix}$$

$$= \begin{bmatrix} 2+10 & -4+15 & 6-20 \\ 6-20 & -8+25 & +15 \\ -4-10 & 15 & -10+20 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 11 & -14 \\ -14 & 17 & 15 \\ -14 & 15 & 10 \end{bmatrix} 3x3$$

2.4.4 Multiplication of matrices: Let A be an m x n matrix, B be n x r matrix. The product of A times B denoted by AB is the m x r matrix whose entry in the ith row and jth column is the sum of product of corresponding elements of ith row of A and jth column of B.

Caution : Observe that the product AB is only defined for matrices A and B such that the number of column of A is same as the number of rows of B. For example if A is 2x3 and B is 3x4, the AB is defined and will be 2x4, however if A is 2x3 and B is 2x4 then AB is not defined.

Self Assessment - II

1. If $A = \begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$. Find AB

2. Find AB and BA, where

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 0 & 1 \\ 5 & 6 & 7 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 6 & 0 & 4 \end{bmatrix}$$

2.5 CRAMMER'S RULE

2.5.1 Crammer's rule for solution of equations linear consider the following set of simultaneous

Equation

$$\begin{array}{cccccccccc} a_{11} & x_1 & + & a_{12} & x_2 & + & a_{13} & x_3 & = & b_1 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & a_{23} & x_3 & = & b_2 \\ a_{31} & x_1 & + & a_{32} & x_2 & + & a_{33} & x_3 & = & b_3 \end{array}$$

Let $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

consider

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, = \begin{vmatrix} a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 & b_1 & a_{12}\,a_{13} \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 & b_2 & a_{22}\,a_{23} \\ a_{31}\,x_1 + a_{32}\,x_x + a_{33}\,x_3 & b_3 & a_{32}\,a_{33} \end{vmatrix}$$

$$= x_1 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, + x_2 \begin{vmatrix} a_{12} & a_{12} & a_{13} \\ a_{22} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix} + x_3 \begin{vmatrix} a_{13} & a_{12} & a_{13} \\ a_{23} & a_{22} & a_{23} \\ a_{33} & a_{32} & a_{33} \end{vmatrix}$$

$$= x_1\,\Delta + 0 + 0$$

$$\therefore x_1 = \frac{\Delta 1}{\Delta} \neq 0$$

$$Similarly\,, x_2 = \frac{\Delta 2}{\Delta}\,, x_3 = \frac{\Delta_3}{\Delta}$$

This is known as Crammer's rule for solving simultaneous linear equations by using determinants.

Note 1: if $\Delta \neq 0$, then the solution to given system is unique.

Note 2 : If $|\Delta| = 0$ and the numerator of all x_1, x_2 x_3 i.e. $|\Delta_1|$, , $|\Delta_2|, |\Delta_3|$ are all zero, then the system has either no solution or infinite solution.

Note 3 : If $|\Delta| =$ O and at least one of $|\Delta_1|, |\Delta_2| |\Delta_3|$ is non-zero, then it has no solution

Ex1: Use the method of determinants to solve the set of equations

$2x_1 \ -x_2 = 3, \ -x_1 + 2x_2 = -3$

Sol. $\underset{2x2}{\overset{A}{\downarrow}} \quad \underset{2x1}{\overset{X}{\downarrow}} = \underset{2x1}{\overset{B}{\downarrow}}$ where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\text{Let } \Delta = [A] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = (2)(2) - (-1)(-1)$$

$$= 4 - 1 = 3 \neq 0$$

$$\Delta_1 = \begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix} = (3)(2) - (-3)(-1)$$

$$= 6 - 3 = 3$$

$$\Delta_2 = \begin{bmatrix} 2 & 3 \\ -1 & -3 \end{bmatrix} = (2)(-3) - (-1)(3)$$

$$= -6 + 3 = -3$$

By Crammer's Rule

$$X_1 = \frac{\Delta_1}{\Delta} = \frac{3}{3} = 1$$

$$X_2 = \frac{\Delta_2}{\Delta} = \frac{-3}{3} = -1$$

Self Assessment-III

1. $2x + y - z = 9$, $x + y + z = 9$, $3x - y - z = -1$

2. $10x + 5y - 5z = 45$, $5x + 5y + 5z = 45$ $15x - 5y - 5z = -5$

2.6 INVERSE OF MATRIX

In order to find an inverse of a matrix, one must know how to find adjoint of a matrix. As inverse of a matrix is symbolically written as :

$$A^{-1} = \frac{adj.(A)}{|A|}$$

2.6.1 Adjoint of a matrix : Let $A = (a_{ij})$ be any square matrix of order n xn, then the transpose of matrix (A_{ij}) where A_{ij} is co-factor of a_{ij} in det. A, is called adjoint of A and is written as adjoint A. Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{then Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Example:

$$\text{if } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}, \text{find Adj.A}$$

Sol. $A_{11} = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix} = 12 - 6 = 6$

$$A_{12} = -\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = -(8-6) = -2, A_{13} = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix} = 6 - 9 = -3$$

$$A_{21} = -\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = -(8-9) = 1, A_{22} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = 4-9 = -5$$

$$A_{23} = -\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = -(3-6) = 3, A_{31} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = 4-9 = -5$$

$$A_{32} = -\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = -(2-6) = 4, A_{33} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 3-4 = -1$$

$$\therefore Adj.A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

2.6.2 Matrix Inversion

The determinant [A] of a 2 x 2 matrix called a second order determinant is derived by taking the product of two elements on the principle diagonal and subtracting from it the product of the two elements of the principle diagonal. Given a general 2 x 2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the determinant is

$$[A] = a_{11}\, a_{22} - a_{12}\, a_{21}$$

The determinant is a single number or scalar and is found only for square matrices. If the determinant of a matrix is equal to zero. The determinant is said to vanish and the matrix is termed as singular. A singular matrix is one in which there exists liner dependence between at least two columns or rows. If $[A] \neq 0$ the matrix A is non-singular and all its rows and columns are linearly independent.

The ranks of a matrix is defined as the maximum number of linearly dependent rows or columns in the matrix. The rank of a matrix also allows for a simple test of linear dependence which follows immediately. Assuming a square matrix of order n

If p (A) = n, A is nonsingular and there is no linear dependence.

If p (A) <n, A is singular and there is linear dependence.

Examples:

i) $A = \begin{bmatrix} 6 & 4 \\ 7 & 9 \end{bmatrix}$ ii) $B = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$

$$
\begin{aligned}
\text{(i)} \quad |A| &= 6\,9 - 4(7) \\
&= 54 - 28 \\
&= 26
\end{aligned}
$$

Since [A] $\neq$ 0, the matrix is non-singular i.e. there is no linear dependence between any of its rows or columns. The rank of A is 2, written ρ (A) = 2

$$
\begin{aligned}
\text{ii)} \quad |B| &= 4(9) - 6(6) \\
&= 36 - 36 \\
&= 0
\end{aligned}
$$

With [B] =0, B is singular and linear dependence exists between its rows and columns. Inspection reveals that row 2 and column 2 are equal to 1.5 times row 1 and column 1 respectively. Hence ρ (B) = 1

Example: Given $A = \begin{bmatrix} 8 & 3 & 2 \\ 6 & 4 & 7 \\ 5 & 4 & 3 \end{bmatrix}$

the determinant [A] is calculated as follows

$$|A| = 8\begin{vmatrix} 4 & 7 \\ 1 & 3 \end{vmatrix} + 3(-1)\begin{vmatrix} 6 & 7 \\ 5 & 3 \end{vmatrix} + 2\begin{vmatrix} 6 & 4 \\ 5 & 1 \end{vmatrix}$$

$$= 8(12-7) - 3(18-35) + 2(6-20)$$

$$= 8(5) + 51 + 2(-14)$$

$$= 40 + 51 - 28$$

$$= 63$$

With $[A] \neq 0$, A is non-singular and ρ (A) = 3

Theorem : If A and B are non-singular square matrices of the same order then $(AB)^{-1} = B^{-1} A^{-1}$

Sol : A & B are non-singular

$$\therefore |A| \neq 0, \quad |B| \neq 0$$

$$\therefore |AB| = |A|.|B| \neq 0$$

$$\therefore A^{-1}, B^{-1}, (AB)^{-1} \text{ exists}$$

$$\therefore AB \text{ is also non-singular}$$

Now $(AB)(B^{-1}. A^{-1}) = A(B B^{-1})A^{-1}$

$$= A I A^{-1} = A A^{-1} = 1$$

Similarly, $(B^{-1} A^{-1})(A B) = 1$

Hence $(A B)^{-1} = B^{-1} A^{-1}$

Example : If $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ find A^{-1} and verify $A^{-1} A = 1$

Sol. $\text{Now } A_{11} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = 3 - 1 = 2, A_{12} = -\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = -(2-1) = -1$

$$A_{13} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 2 - 3 = -1, A_{21} = -\begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix} = -(2-5) = +3$$

$$A_{22} = \begin{bmatrix} 1 & 5 \\ 1 & 1 \end{bmatrix} = 1 - 5 = -4, A_{23} = -\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = -(1-2) = 1$$

$$A_{31} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13, A_{32} = -\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} = -(1-10) = 9$$

$$A_{33} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 3 - 4 = -1$$

$$\text{Adj. } A = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}^T = \begin{vmatrix} 2 & -1 & -1 \\ 3 & -4 & 1 \\ -13 & 9 & -1 \end{vmatrix}^T$$

$$= \begin{vmatrix} 2 & 3 & -13 \\ -1 & -4 & 9 \\ -1 & 1 & -1 \end{vmatrix}$$

Also $[A] = a_{11} A_{11} + a_{12} A_{12} + A{=}a_{13} A_{13}$

$= (1)\ (2) + 2\ (-1) + 5\ (-1)$

$= 2 - 2 - 5$

$= -5$

$$A^{-1} = \left[\frac{1}{|A|}\right] \text{Adj.} A = -\frac{1}{5}\begin{bmatrix} 2 & 3 & -13 \\ -1 & -4 & 9 \\ -1 & 1 & -1 \end{bmatrix}$$

$$A^{-1}A(-1/5)\begin{bmatrix} 2 & 3 & -13 \\ -1 & -4 & 9 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = -\frac{1}{5}(-5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A^{-1} A = 1

Self Assessment - IV

1. Find the inverse of a matrix A. If A =

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$$

2. Find the adjoint of the Matrix A. If

$$A = \begin{bmatrix} 8 & 3 \\ 6 & 4 \end{bmatrix}$$

2.7 ECONOMICS APPLICATION OF MATRICES

Example: Given y = C + I_0, where C = C_0 + b y use matrix inversion to find the equilibrium level of y and c.

Solution: The given equation can first be rearranged so that the endogenous variables c and y, together with their co-efficient, are on the left hand side of the equation and exogenous variables C_0 and I_0 are on the right.

$$y - c = I_0$$

$$-by + c = c_0$$

Thus, $\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} y \\ c \end{bmatrix} = \begin{bmatrix} I_0 \\ c_0 \end{bmatrix}$

The determinant of the co-efficient matrix is [A] = 1 (1) - (-b) (-1)

= 1 - b

The co-factor matrix is

$$C = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}$$

$$\text{Adj.A} = \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

$$\text{and } A^{-1} = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

$$\text{letting } x = \begin{bmatrix} y \\ c \end{bmatrix}, \quad x = \frac{1}{1-b} \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} i_0 \\ c_0 \end{bmatrix}$$

$$= \frac{1}{1-b} \begin{bmatrix} I_0 + c_0 \\ bI_0 + c_0 \end{bmatrix}$$

Thus, $y = \frac{1}{1-b}(I_0 + c_0), c = \frac{1}{1-b}(c_0 + bI_0)$

Example: Use crammer's rule to solve for x and y when the cost function $C = 8x^2 - xy + 12y^2$ and the firm is bound by λ contract to produce a minimum combination of goods totaling 42 that is, subject to the constrain x + y= 42.

Set the constraint equal to zero, multiply it by and form the lagrangian function

$$C = 8x^2 - xy + 12y^2 + \lambda(42 - x - y)$$

$$\frac{\partial TC}{\partial x} = 16x - y - \lambda = 0$$

$$\frac{\partial TC}{\partial y} = -x + 24y - \lambda = 0$$

$$\frac{\partial TC}{\partial \lambda} = 42 - x - y = 0$$

Re-arrange the equations

$16x - y - \lambda = 0$

$\rightarrow$ $X + 24y - \lambda = 0$

$\rightarrow$ $X - y = -42$

and set them in matrix form

$$\begin{bmatrix} 16 & -1 & -1 \\ -1 & 24 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -42 \end{bmatrix}$$

expanding from third column

$$|A| = (-1)(1+24) - (-1)(-16-1) + 0$$

$$= -25 - 17 = -42$$

$$A_3 = \begin{bmatrix} -16 & -1 & 0 \\ -1 & 24 & 0 \\ -1 & -1 & -42 \end{bmatrix}$$

$$[A_1] = -42(1+24) = -1050$$

$$A_2 = \begin{bmatrix} -16 & 0 & -1 \\ -1 & 0 & -1 \\ -1 & -42 & 0 \end{bmatrix}$$

Expanding along the second column

,

$$[A_2] = -(-42)(-16-1) = -714$$

$$A_3 = \begin{bmatrix} -16 & -1 & 0 \\ -1 & 24 & 0 \\ -1 & -1 & -42 \end{bmatrix}$$

Expanding along the third column

$$[A_3] = -42(384-1)$$

$$= -16,\ 086$$

Thus, x $= \dfrac{|A_1|}{|A|} = -\dfrac{1050}{-42} = 25$

y $= \dfrac{|A_2|}{|A|} = -\dfrac{714}{-42} = 17$

$$\text{and}\,\lambda = \frac{|A_3|}{|A|} = -\frac{16086}{-42} = 383$$

2.8 SUMMARY

We end this lesson by summarizing what we have covered in it.

i) Defined Matrix and types of matrices.

ii) Used crammer's rule to solve simultaneous equations.

iii) Rank of matrix and matrix inversion.

2.9 LESSON END EXERCISE

Q1. Define and give example of each of the following :-

i) Trace of a matrix ii) Identify matrix

Q2. If $A = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 1 & 5 \\ 3 & -2 & -1 \end{bmatrix}$

Find a square matrix X of the order 3x3 such that A+B+X =0

Q3. Prove by an example that AB can be zero matrix when neither of A+B is zero matrix.

Q4. Explain with illustration

i) Transpose of a matrix ii) Symmetric matrix

iii) Skew Symmetric matrix

Q5. Solve by Crammer's rule

x+2y+3z=1; 2x+2y+4z=2; 3x+4y+3z=3

Q6. Consider the following national income determination model:-

Y = C+I+G

C = a+b (Y-t)

T = d+ t Y

Where Y (national income), C consumption expenditure, T (Tax collection) are endogenous variable; I (investment) and G (government expenditure) are exogenous variables; t is income tax rate. Solve for endogenous variables, using Crammer's rule.

Q7. If

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find Adj.A

Q8. Find the inverse of $\begin{bmatrix} 1 & 3 & -2 \\ 5 & 0 & 6 \\ 9 & -2 & 7 \end{bmatrix}$

Q9. For the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, verify

$AA' = I = A'A$, A' is transpose of A

Q10. Explain Crammer's rule for solving three equations.

2.9 SUGGESTED READING

Aggarwal, C.S & R.C. Joshi : Mathematics for Students of Economics (New Academic Publishing Co).

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QUADRATIC EQUATIONS, EIGEN ROOTS AND EIGEN VECTORS, FUNCTIONS : LINEAR AND NON-LINEAR, CONVEX AND CONCAVE

STRUCTURE

3.1 INTRODUCTION

Quadratic equations are very popular in economic analysis. The average cost curves, cost curves, marginal cost curves and average variable cost curves are represented by quadratic equations. Demand curve is also of quadratic form. Quadratic equations are indispensable for the students of economics.

3.2 OBJECTIVE

In this lesson, our effort is to help the students to solve the quadratic equations with given techniques. After studying this lesson, you should be able to

- Solve quadratic equations
- Find the roots of the quadratic equation
- Eigen roots and eighteen vectors
- Different types of functions i.e. linear and non-linear, convex and concave.

3.3 DEFINITION QUADRATIC EQUATIONS

An equation of the type $ax^2+bx+c = 0$ where a, b, c are constants, x is unknown quantity is known as quadratic equation. Thus, $3x^2+6x+7 =0$, $5x^2+9=0$, $x^2+3x+1=0$ etc are all quadratic equations. The equation $ax^2+bx+c=0$, where a, b, c are constants, is called standard quadratic equation. Here 'a' is the co-efficient of x^2, b of x and c, the constant term.

Definition Roots of quadratic equations

The values of the unknown quantities which satisfies the given equations are called the roots of that equations

For example $2x + 6 = 0$, is satisfied by $x = -3$

$\therefore$ $- 3$ is the root of $2x + 6 = 0$

How to solve standard quadratic equation

$$ax^2 + bx + c = 0$$

Sol. We take the constant term to the other side

$$ax^2 + bx = -c$$

Divide both sides of the equation by a (co-efficient of x^2), we get

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

Add $\left(\frac{b}{2a}\right)^2$ to both sides $\left\{\frac{1}{2}\text{co}-\text{eff.of } x\right\}^2$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left[x + \frac{b}{a}\right]^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\frac{b^2 - 4ac}{4a^2}$$

Taking square root, we get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$= = \pm\sqrt{\frac{b^2 - 4ac}{2a}}$$

$$\therefore x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

$$\frac{= -b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note 1 : The two values of x are called the roots of the quadratic equation $ax^2+bx+c=0$ and are denoted by Greek letters α (Alpha) and β (Beta). Thus,

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Ex. 1

Solve $8x - 2 = \frac{3}{x}$

$8x^2 - 2x = 3$

$8x^2-2x-3 = 0$

Compare with $ax^2 + bx + c = 0$

$a = 8,\ b = -2,\ c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(8)(-3)}}{2x8}$$

$$= \frac{2 \pm \sqrt{4+96}}{16}$$

$$= \frac{2 \pm 10}{16}$$

$$= \frac{12}{16}, -\frac{8}{16}$$

$$= \frac{3}{4}, -\frac{1}{2}$$

The equilibrium price of a commodity is determined at the point where the quantity demanded is equal to quantity supplied. Therefore by solving equation (i) and (ii) simultaneously, we can find equilibrium price and quantity from (1)

$$p = \frac{8-3q}{2}$$

Substitute in (ii)

$$\left(\frac{8-3q}{2}\right)^2 + 2\left(\frac{8-3q}{2}\right)q + 3q^2 = 17$$

$$\frac{64-48q+9q^2}{4} + \frac{8q-3q^2}{2} + 3q^2 = 17$$

$$64 + 9q^2 - 48q + 32q = 68$$

$$9q^2 - 16q - 4 = 0$$

$$q = \frac{16 \pm \sqrt{256-4(9)(-4)}}{2x9}$$

$$= \frac{16 \pm \sqrt{256+144}}{18}$$

$$= \frac{16 \pm \sqrt{400}}{18}$$

$$= \frac{16 \pm 20}{18}$$

$= 2, -2/9$

Rejecting negative as quantity bought or sold cannot be negative

$\therefore$ q = 2, $\therefore$ p $= \frac{8-3(2)}{2} = \frac{8-6}{2} = 1$

$\therefore$ Equilibrium price = 1, equilibrium quantity = 2

Ex.2 : The marginal cost curve of a firm under perfect competition is given as MC = q^2-8q-1, if the market price if fixed at Rs. 19 per unit, find the equilibrium of output.

Solution: Given

$MC = q^2 - 8q - 1$, and price = 19

We know under perfect competition, a firm is in equilibrium when MC=price

$\therefore\ q^2 - 8q = 19$

$q^2 - 8q - 20 = 0$

$$q = \frac{8 \pm \sqrt{64 + 80}}{2}$$

$$= \frac{8 \pm \sqrt{124}}{2}$$

$$= \frac{8 \pm 12}{2}$$

= 10, -2

But $q \neq -2$, $q = 10$

and equilibrium level of output = 10

Self-Assessment - I

1. The demand for good of an industry is given by the wequation pq = 100, where p stands for price and q for quantity demanded. Supply is given by the equation 20 - 3p = q. What is the equilibrium price and quantity demanded.

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3.4 EIGEN ROOTS AND EIGEN VECTORS

Given a square matrix A, if it is possible to find a vector $V \neq 0$, and a scalar c such that

$$AV = cV \qquad \text{--- (i)}$$

The scalar c is called the characteristic root, latent root or eigen value and the vector sis called the characteristic vector4, latent vector or eigen vector. Equation (i) can be expressed

$$AV = c\,IV \qquad \text{--- (ii)}$$

where A - cI is called the characteristic matrix of A. Since by assumption $V \neq 0$, the characteristics matrix must be singular and thus its determinant must vanish.

If A = 3x3 matrix, then

$$|A - cI| = \begin{vmatrix} a_{11} - c & a_{12} & a_{13} \\ a_{23} & a_{22} - c & a_{23} \\ a_{31} & a_{32} & a_{33} - c \end{vmatrix} = 0$$

with $|A - cI| = 0$ in (ii), there will be an infinite number of solutions for V. To force a unique solution, the solution may be normalized by requiring of the elements of v_i of V that $\sum v_i^2 = 1$

if

1) All characteristic roots (c) are positive A is, positive definite.

2) All c's are negative, A is definite negative.

3) All c's are non-negatives and at least one c=0, A is positive semi definite.

4) All c's are non-positive and at least one c=0, A, is negative semi definite.

5) Some c's are positive and others negative, A is sign indefinite.

Ex. 1: Use Eigen values to determine sign definiteness for

$$A = \begin{bmatrix} 10 & 3 \\ 3 & 4 \end{bmatrix}$$

To find the characteristic roots of A, the determinant of the characteristic matrix A-cI must equal zero. Thus

$$|A - cI| = \begin{vmatrix} 10-c & 3 \\ 3 & 4-c \end{vmatrix} = 0$$

(10-c) (4-c) - 9 = 0

$40\text{-}10c\text{-}4c + c^2 - 9 = 0$

$c^2 - 14c + 31 = 0$

Using the quadratic formula

$$c = \frac{14 \pm \sqrt{196 - 4(31)}}{2}$$

$$= \frac{14 \pm \sqrt{72}}{2}$$

$$= \frac{14 \pm 8.485}{2}$$

$$= \frac{22.848}{2}, \frac{5.515}{2}$$

= 11.424 2.7575

$c_1 = 11.424$, $c_2 = 2.7575$

with both characteristic roots positive A is positive definite

Ex.2: $A = \begin{bmatrix} 4 & 6 & 3 \\ 0 & 2 & 5 \\ 0 & 1 & 3 \end{bmatrix}$

Sol. $$|A - cI| = \begin{bmatrix} 6-c & 1 & 0 \\ 13 & 4-c & 0 \\ 5 & 1 & 9-c \end{bmatrix} = 0$$

Expanding from third column

$$|A - cI| = (9-c)[(6-c)(4-c) - 13] = 0$$

$$= (9-c)\left[24 - 6c - 4c + c^2 - 13\right] = 0$$

$$= (9-c)\left[11 - 10c + c^2\right]$$

$99 - 90c + 9c^2 - 11c + 10c^2 - c^3 = 0$

$-c^3 + 19c^2 - 101c + 99 = 0$

will equal zero if

$9 - c = 0$ or $(6 - c)(4-c) - 13 = 0$

$c^1 = 0$ $24 - 6c - 4c + c^2 - 13 = 0$

$c^2 - 10c + 11 = 0$

$$c = \frac{10 \pm \sqrt{100 - 4(11)}}{2}$$

$$= \frac{10 \pm 7.48}{2}$$

$= 8.74, 1.26$

$c_2 = 8.74$, $c_3 = 1.26$

with all the latent roots positive, A is positive definite

Ex. 3 :

Given $A = \begin{bmatrix} 6 & 6 \\ 6 & -3 \end{bmatrix}$

Find (a) the characteristic roots and

(b) the characteristic vectors

Sol.

a) $|A - cI| = \begin{bmatrix} 6-c & 6 \\ 6 & -3-c \end{bmatrix} = 0$

(6-c) (-3-c) - 36 = 0

$-18 - 6c - 3c + c^2 - 36 = 0$

$c^2 - 9c - 54 = 0$

(c-9) (c+6) = 0

$c_1 = 9,\ c_2 = -6$

with one root positive and other negative, A is sign indefinite

b) Using $c_1 = 9$, for the first characteristic vector V_1

$$\begin{bmatrix} 6-c & 6 \\ 6 & -3-9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Normalizing

$$(2v_2)^2 + v_2^2 = 1$$

$$5v_2^2 = 1$$

$$V_2 = \sqrt{0.2}$$

$$V_1 = 2v_2 = \sqrt[2]{0.2}$$

Thus $$V_1 = \begin{bmatrix} \sqrt[2]{0.2} \\ \overline{\sqrt{0.2}} \end{bmatrix}$$

Using c_2 = -6 for the second characteristics vector

$$\begin{bmatrix} 6-(-6) & 6 \\ 6 & -3-(-6) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Normalizing

$$v_1^2 \;+\; (-2v_1)^2 = 1$$

$$v_1^2 \;+\; 4v_1^{\,2} = 1$$

$$5v_1^{\,2} = 1$$

$$v_1 = \sqrt{115} = \sqrt{0.2}$$

$$v_2 = \sqrt[-2]{0.2}$$

Thus $$v_2 = \begin{bmatrix} \sqrt{0.2} \\ \sqrt[-2]{0.2} \end{bmatrix}$$

Self-Assessment - II

1. If $A = \begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$, find a> eigen roots and, b> eigen vectors.

2. If $A = \begin{bmatrix} 3 & 3 \\ 3 & -6 \end{bmatrix}$, find eigen vectors.

3.5 LINEAR FUNCTION :

Generally, the functions of the form, $Q = A + bP$ are called linear because of the fact that their graphs always turn out to be straight lines.

Functions of the form

$Q = a + bP + cY + d\pi + cT$ are also called linear, since if all but two of the variables Q, P, Y, π and T are called constant the relation between the remaining two always produces a straight line as its graph if any of the variables which has assumed constant were to change, this would alter the relation between any of the others.

Non-linear functions : To appreciate what is implied by linear function, consider other possible functions. The demand curve showing the relation between Q and P, for given Y, π and T, could have been curved. Such a function is said to be non-linear function or curvilinear.

3.6 CONCAVITY AND CONVEXITY :

A function f (x) is concave at x = a if in small region close to the point [a, f(a)] the graph of the function lies completely below its tangent line. A function is convex at x = a, if in an area very close to [a, f(a)] the graph of the function lies completely above its tangent line. A positive second derivative at x = a, denotes the function is convex at x = a and negative second derivative at x =a denotes the function is concave at a.

Test to see if the function is convex or concave.

a) $y = -2x^3 + 4x^2 + 9x - 15$ at $x = 3$

$$\frac{dy}{dx} = y' = -6x^2 + 8x + 9$$

$$\frac{d^2y}{dx^2} = y'' = -12x + 8 = -12(3) + 8 = -36 + 8 < 0 \text{ concave}$$

b) $y = (5x^2-8)^2$ at $x = 3$

$$\frac{dy}{dx} = 2(5x^2 - 8)\,10x = 20x(5x^2 - 8) = 100x^3 - 160x$$

$$\frac{d^2y}{dx^2} = 300x^2 - 160$$

$= 300(9) - 160$ at $x = 3$

$= 2540 > 0$ convex

Self-Assessment - III

1. Define linear function and non-linear function.

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2. What do you mean by concavity and convexity ?

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3.7 SUMMARY

In this lesson we have explained

- Quadratic equation
- Eigen roots and Eigen vectors
- Linear and non-linear functions
- Convex and concave function

3.8 LESSON END EXERCISE

i) Given the demand law, $p = 85-4q-q^2$. Find the amount demanded (q) when p= 40. If the price rises to 64, how much will the demand contract.

ii) The demand and supply equations are given by $p - q = 1$ & $p^2 + q^2 = 25$; where p & q stand for price and quantity respectively. Find equilibrium price and quantity

iii) Discuss the nature of roots of the following equations

$x^2-4x + 4=0$

$3x^2+5x+7 = 0$

iv) Define characteristic vectors. Find the characteristic roots and the characteristics vectors of the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

3.9 SUGGESTED READINGS

Aggarwal,C.S & R.C.Joshi: Mathematics for Students of Economics (New Academic Publishing co)

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M.A. Economics		**Lesson No. 4**
C.No. 103	**Semester - 1st**	**Unit I**

CONCEPT OF SEQUENCE, LIMIT OF SEQUENCE CONCEPT OF LIMIT AND CONTINUITY, ECONOMIC EXAMPLES AND APPLICATIONS

STRUCTURE

4.1 INTRODUCTION

Calculas an important branch of mathematics with a wide range of application of based, in general, on the idea of a limit what gives calculus its power and distinguishes it from algebra is the concept of limit. Sometimes, we are interested in the limiting behaviour of a function as the independent variable approaches a finite or infinite value. We may be interested in determining the limiting saturation levels of sales or profit when promotional effort is increased or the limiting value of learning as the number of hours of study increases.

4.2 OBJECTIVES :

After reading this lesson you would be able to understand

- Concept of sequence.
- Limit of sequence
- Concept of limit and continuity
- Economic application of the limit.

4.3 SEQUENCE :

Let $y = \frac{1}{2x}$, where

Y is a single valued function of x. To each other value of x, there corresponds one and only one value of y. Let us pose a question. How does the function behave as a sequence of values are allotted to x according to some law? From the above function, we get

x : 1 2 3 4 5 _ _ 1000 _ _ _.

y : 1/2 1/4 1/6 1/8 1/10 _ _ 1/2000 _ _ _.

To the x-sequence, there corresponds a y-sequence. Do we get an idea about the behaviour of the function from the y-sequence has been constructed according to some value. It is not a collection of some arbitrary numbers. The idea one gets is that as x becomes larger and larger, u or f (x), becomes smaller and smaller. Let us go ahead and say that as x tends to infinity, y tends to zero. We cannot make it equal to zero by making x larger and larger, but we can make it very close to zero.

Let us consider another function

$Y = 3 - 1/x$

The values of y will behave as

x : 1 2 3 4 5 _ _ 1000 _ _ _.

y : 1/2 1/4 1/6 1/8 1/10 _ _ 1/2000 _ _ _.

Thus, as we assign larger and larger positive values to x the corresponding y - sequence or the function y = 3 -1/x behave in such a way that it gets closer and closer to a limit 3. Let us form the sequence of negative integral of x, and see how the function 3 - 1/x behaves

x : -1 -2 -3 _ _ -1000 _ _ - 1000

y : 4 7/2 10/3 _ _ 301/100 _ _3100/1000

The above pair of resulted sequence shows that x tends to minus infinity, the function y = 3 -1/x tends to the limit 3.

In this way, we can establish x - sequence and the corresponding y sequence for other function such as y = 1/x+3, $y = 2 \pm (-1)^X$, $y = x^2+x+1$ and so on these different pairs of associated sequences lead to some interesting possibilities. Thus

i) f(x) tend to L as x tends to infinity. In symbols

Lt f(x) = L, where L is finite

$x \to \infty$

ii) f(x) tends to infinity as x tends to infinity. In symbols

Lt f (x) = ∞

$x \to \infty$

iii) f(x) tends to minus infinity as x tends to infinity. In symbols

Lt f(x) = ∞

$x \to \infty$

iv) f(x) tends to limit L, as x approaches i.e. Ltf (x)=L, Where C ± L are

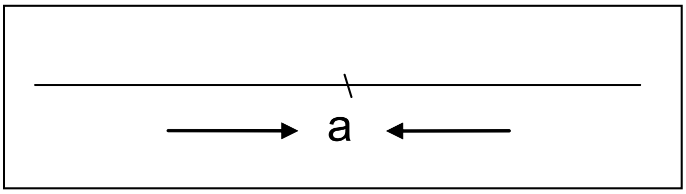
both $x \to \infty$ finite.

4.4 THE CONCEPT OF LIMIT

Let y=f(x), be a function of x. Limits describe as to what happens to a function f(x) as its variable x approaches to a particulars value say 'a'.

A variable x is said to approach a particular value 'a' if x takes a succession of values, nearer and nearer (but never equal to) a i.e. each, succeeding value and there is no end to this dynamic process of coming nearer and nearer to 'a' in this process, the absolute value $|x-a|$ becomes smaller and smaller can be made as small as we like and denote this number by ε (read as epsilon)

Where ε is any arbitrary number which may be as small as we please



Form the figure, it is clear that x can approach 'a' either from the left hand side of right hand side.

When x approaches 'a' from the left, x always remain less than 'a' but x-a becomes smaller and smaller. We denote this fact by writing $x \to a$ - or $x \to \bar{a}$

When x approaches 'a' from the right, x always remains greater than 'a'. We denote this fact by writing 'x' $\to a^+$ or $x \to a^-$

4.4.1 Limit of a function

A function f(x) is said to tend to the limit 'I' as x tends to A. i.e. Lim = 1,

$x \to a$

If given any positive number ε, however, small

We can find a number δ (depending on $\in$ such that $[f(x) - 1| < \in$ whenever

$|x-a| < \delta$. Thus

$\lim\limits_{x \to a} f(x) = 1$ if $|f(x)-1| <$ whenever $|x-a| < \delta$

Theorems on limits

Let $f_1(x)$ and $f_2(x)$ be any two functions of x and k and a be any constants then.

Th_1 The limit of a constant is the constant itself, i.e,

$$\lim_{x\to a} K = K$$

Th_2, The limit of the product of a constant and a function is equal to the product of the constant and limit of the function i.e.

$$\lim_{x\to a} [K, f(x)] = K [\lim_{x\to a} f_1(x)]$$

Th_3, The limit of the sum of two (or more) function is equal to the sum of their limits i.e.

$$\lim_{x\to a} [f_1(x)+f_2(x)] = \lim_{x\to a} f_1(x)+ \lim_{x\to a} f_2(x)$$

Th_4, The limit of the different of two (or more functions is equal to the difference of their limits, i.e.

$$\lim_{x\to a} [f_1(x)-f_2(x)] = \lim_{x\to a} f_1(x) - \lim_{x\to a} f_2(x)$$

Th_5, The limit of the product of two functions is equal to the product of their limits, i.e.,

$$\lim_{x\to a} [f_1(x)\, f_2(x)] = [\lim_{x\to a} f_1(x)]\, [\lim_{x\to a} f_2(x)]$$

Th_6, The limit of the quotient of two function is equal to the quotient of their limits provided the limits of the quotient is not zero i.e.

$$\lim_{x\to a} \left[\frac{f_1(x)}{f_2(x)}\right] = \lim_{x\to a} f_1(x) \div \lim_{x\to a} f_2(x)$$

provided the limit of the denominator is not zero i.e. $\lim_{x\to a} f_2(x) \neq o$

4.4.2 Rules to find the limit of a function

1) For finding $\lim_{x \to a} f(x)$, put x=a directly

in the expression of f(x), provided we do not get a form of the type 0/0 or a form in which denominator is 0 and hence find the limiting values.

2) In $\lim_{x \to a} f(x)$, if the expression of f(x), after putting x = a, attains the form 0/0 or a form in which denominator is 0 then put x=a+h, h being a small number so that when $x \to a$, $h \to 0$ and $\lim_{x \to a} f(x) = \lim_{h \to a} (a+h)$

Ex.1 Evaluate i) $\lim_{x \to 2} (x^2-5x+6)$

ii) $\lim_{x \to 0} (x^2-1)$

iii) $\lim_{x \to 0} \sqrt{x}$

Solution :

i) $\lim_{x \to 2} (x^2-5x+6) = \lim_{x \to 2} x^2 - \lim_{x \to 2} 5x + \lim_{x \to 2} (6)$

$= 2^2-5(2)+6$

$= 4-10+6$

$= 0$

ii) $\lim_{x \to 1} (x^2-1) = \lim_{x \to 1} (x^2) - \lim_{x \to 1} 1$

$= 1^2-1 = 0$

Or

$\lim_{x \to 1} (x^2-1) = \lim_{x \to 0} [(1+h)^2-1]$

$= \lim_{x \to 6} [1+2h+h^2-1]$

$$= \lim_{x\to 0} 2h+h^2$$

$$= (0)^2+2(0)$$

$$= 0$$

iii) $\lim_{x\to 0} \sqrt{x} = 0$ when $x \to 0^+$ but becomes non-real $\therefore$ Lim does'nt exist

Evaluate :

(i) $$\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{\lim_{x\to 2} x^2-4}{\lim_{x\to 0} x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}$$

which is undefined so in such question, we proceed as

$$\text{Lim } \frac{x^2-4}{x-2} = \lim_{x\to 0}\left[\frac{(2+h)^2-4}{(2+h)-2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(4+h^2+4h-4}{2+h-2}\right]$$

$$= \lim_{x\to 0}\left[\frac{h^2+4h}{h}\right]$$

$$= \lim_{x\to 0}\left[\frac{h(h+4)}{h}\right]$$

$$= \lim_{x\to 0}(h+4)$$

$$= 4$$

ii) $$\lim_{x\to 2}\frac{(x^2-5x+6)}{x^2+x-6} = \frac{\lim_{x\to 2}(x^2-5x+6)}{\lim_{x\to 2}(x^2+x-6)} = \frac{4-10+6}{4+2-6} = \frac{0}{0}$$

which is undefined so we proceed as follows

$$\lim_{x\to 2}\frac{(x^2-5x+6)}{(x^2+x-1)}=\lim_{x\to 2}\left[\frac{(x-3)(x-2)}{(x+3)(x-2)}\right]=\lim_{x\to 2}\frac{(x-3)}{(x+3)}=-\frac{1}{5}$$

iii) $$\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a}}{x}=\frac{\sqrt{a+0}-\sqrt{a}}{0}=\frac{0}{0}=\text{undefined}$$

∴ In such questions, we would rationalize first and then find the limit.

$$\lim_{x\to 0}\frac{\sqrt{a+x}-\sqrt{a}}{x}=\lim_{x\to 0}\left[\frac{\sqrt{a+x}-\sqrt{a}}{x}\times\frac{\sqrt{a+x}+\sqrt{a}}{\sqrt{a+x}+\sqrt{a}}\right]$$

$$=\lim_{x\to 0}\left[\frac{x}{x(\sqrt{a+x}+\sqrt{a}}\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{\sqrt{a+x}+\sqrt{a}}\right]$$

$$=\frac{1}{\sqrt{a}+\sqrt{a}}$$

$$=\frac{1}{2\sqrt{a}}$$

Rule to find the limit when $x\to\infty$ or $x\to -\infty$

In question involving limits, we first divide the numerator and the denominator by the highest power of x in the denominator and then evaluate the limit as $x\to\infty$ or $x\to -\infty$

Example : Evaluate $=\lim_{x\to\infty}\left(\frac{4x^2+x-2}{2-x+6x^2}\right)$

Sol. If we directly put the limit, we get ∞/∞ which is undefined. So we divided the function by the highest power of x in the denominator, vix. x^2

$$\therefore \; = \lim_{x\to\infty}\left[\frac{4x^2+x-2}{2-x+6x^2}\right] = \lim_{x\to\infty}\left[\frac{4+\frac{1}{x}+\frac{2}{x^2}}{\frac{2}{x^2}-\frac{1}{x}+6}\right]$$

As $x \to \infty$, $1/x \to 0$

$$= \frac{4+0-0}{0-0+6}$$

$$= \frac{4}{6} = \frac{2}{3}$$

Example : Prove that $= \lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Sol. $\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty}\left[(1+n.\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+.....\right]$$

[By Binomial Theorem]

$$= \lim_{x\to\infty}\left[1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!}\right]$$

$$= 1+1+1-0\left(\frac{1}{2!}\right)+(1-0)(1-0)\frac{1}{3.}+...+\infty$$

$$= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....\infty$$

$= e.$

Example : Prove that $= \underset{x \to \infty}{\text{Lim}} \frac{(1+x)^n - 1}{x} = n$

Sol. We Know that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{L2} x^2 + ...$$

[By Bionomial Theorem]

$$\therefore \underset{x \to \infty}{\text{Lim}} \left[\frac{(1+x)^n - 1}{x} \right] = \underset{x \to \infty}{\text{Lim}} \left\{ x \frac{\left[1 + nx + \frac{n(n-1)}{2!} x^2 + ... \right] - 1}{x} \right\}$$

$$= \underset{x \to \infty}{\text{Lim}} \left\{ \frac{nx + \frac{n(n-1)}{2} x^2 + ...}{x} \right\}$$

$$= \underset{x \to \infty}{\text{Lim}} \left\{ \frac{n + \frac{n(n-1)}{2} x^2 + ...}{x} \right\}$$

$$= n + 0 + 0$$

$$= n$$

Example : Prove that $= \underset{x \to a}{\text{Lim}} \frac{x^n - a^n}{x - a} = na^{n-1}$, $a > 0$ and n is any rational nos.

Solution : Let x=a+h, so that as $x \longrightarrow a$, $h \longrightarrow 0$

$$\underset{x\to a}{\mathrm{Lim}}\frac{x^n-a^n}{x-a}=\underset{h\to 0}{\mathrm{Lim}}\left[\frac{(a+h)^n-a^n}{(a+h)-a}\right]=\underset{h\to 0}{\mathrm{Lim}}\left[\frac{(a+h)^n-a^n}{m}\right]$$

$$=\underset{h\to 0}{\mathrm{Lim}}\left[\frac{a^n(1+h/a)^n-a^n}{h}\right]$$

$$=\underset{h\to 0}{\mathrm{Lim}}\left[\frac{a^n(1+n.h/a+\frac{n(n-1)(h/a)^2}{2!}+a^n}{h}\right]$$

$$=\underset{h\to 0}{\mathrm{Lim}}\left[\frac{a^n+a^n.n.h./a^n\frac{n(n-1)(h/a)^2}{2!}+....}{h}\right]-a^n$$

$$=\underset{h\to 0}{\mathrm{Limh}}\left[\frac{n.a^{n-1}+\frac{n(n-1)}{2!}a^{n-2}h+\text{ terms of higher power of h}}{h}\right]^n$$

$$=\underset{h\to 0}{\mathrm{Limh}}\left[\frac{n.a^{n-1}+n(n-1)a^{n-2}.h+\text{ terms of higher power of h}}{h}\right]^n$$

Cancelling h as $h \to 0$

$$=n.a^{n-1}+0+0$$

Self-Assessment - I

1. Evaluate the following :

a) $\lim_{x \to 0} (2 + 3x + x^2)$

b) $\lim_{x \to 1} \left(\frac{x^2 - 1}{x^2 + 1} \right)$

c) $\lim_{x \to 3} \sqrt{5 - x}$

4.4.4. Continuity of a function

A function of f(x) is said to be continuous at a point x a, if

i) f(a), the value of the function f(x) at x = a is defined.

ii) Lim.f(x) exists, i.e.

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \text{ or } f(a + 0) = f(a - 0)$$

iii) Value of the function at x=a=limit of the function at x=a, i.e.

f(a) = $\lim_{x \to a} f(x)$

In short, for function to be continuous limit and value should both exist and be equal to each other.

Continuity in an interval

A function f(x) is said to be continuous in an interval (a,b) or a<x<b if it is continuous at every point of the interval. Since a and b are lower and upper ends of the interval f(x) is continuous at x = a, if

f(a) = $\lim_{x \to a+0} f(x + 0)$ and continuous at x=b if

$$f(b) = \lim_{x \to b+0} f(x) = f(b - 0)$$

Definition of a continuous function at a point x = a, if, corresponding to any arbitrary assigned positive number $\in$, however small (but not equal to zero), there exists a positive number δ, such that

$$| f(x) - f(a) | < \in$$

If such a δ, as defined, cannot be found then the function is said to be discontinuous at x = a

Discontinuous function

A function is said to be discontinuous at x = a, if any of the following three conditions happens

i) Value of the function at x = a, i.e. f(a) does not exist.

ii) Limit of the function at x = a doesn't exist

i.e. $\lim_{x \to a+} f(x) \neq \lim_{x \to a-} f(x)$ or $f(a+0) \neq f(a-0)$

iii) Value of limit, at x = a i.e.

$$f(a) \neq \lim_{x \to 0} f(x)$$

Example : Examine the continuity of the function f(x) at x = 1 and x = 2

If $f(x) = x^2$ when $0 < x < 1$

$= x$ when $1 \leq x < 2$

$= x^3/4$ when $2 \leq x < 3$

At x=1, f(x)=x $\therefore$ f(1) = 1

Also $\lim_{x \to 1^+} [f(x)] = \lim_{x \to 1^+} (x)^2 = 1$

$$\lim_{x \to 1^-} [f(x)] = \lim_{x \to 1^+} (x)^2 = 1$$

Since $\lim_{x \to 1^+} [f(x)] = \lim_{x \to 1-} f(x) = 1 = f(1)$ [Value of function]

i.e. Limit of the function = Value of the function

Hence, the function f(x) is continuous at x=1

At x=2, $f(x) = x^3/4$ $\therefore$ $f(2) = 2^3/4 = 2$

Also $$\lim_{x\to 2+}[f(x)] = \lim_{x\to 2+}\left[\frac{x^3}{4}\right] = \frac{2^3}{4} = 2$$

$$\lim_{x\to 2-}[f(x)] = \lim_{x\to 2-}\left[\frac{x^3}{4}\right] = 2$$

Since limit of function = Value of function

Hence the function is continuous at x=2

Example : Find the points of discontinuity of function n

$f(x) = 2x^2+6x-5 / 12x^2+x-20$

Sol. The given function f(x) is undefined at the points where the denominator viz. $12x^2+x-2$ is zero

$\therefore$ The points of discontinuity of the function are given by the solution of $12x^2 + x - 20 = 0$

$$x = \frac{-1 \pm \sqrt{1-4(12)(-20)}}{2(12)}$$

$$= \frac{1 \pm \sqrt{1-960}}{24}$$

$$= \frac{-1 \pm 31}{24} = \frac{-32}{24}, \frac{36}{24}$$

$$= \frac{-4}{3}, \frac{5}{4}$$

Hence the points of discontinuity of the given function f(x) are x =-4/3 and x=5/4

4.5 ECONOMIC APPLICATION

Example 1 : If MR=AR (1-1/e), where e is the price elasticity of the demand, find the limit of and interpret the phenomenon

Sol. We have to find the $\lim_{x \to \infty}$ MR. We know that $e \to \infty$, $1/e \to 0$. Also AR is constant

$$\therefore \lim_{x \to \infty} MR = \lim_{e \to \infty}\left[AR(1-\frac{1}{e})\right]$$
$$= AR[1-0]$$
$$= AR$$

Which in economics, means that when the demand is perfectly elastic ($e \to \infty$) then MR and AR are equal.

Example 2 : The consumption expenditure (C_E) as a function of personal income (Y) is given by C_E = 100+0.7Y ii) C_E = 150+0.84. Find the limit of saving S where S = Y - C_E. When Y tends to 500

Solution : We know S = Y - C_E

i) Since C_E = 100+0.74

$\therefore$ S = Y - (100+0.7Y)

= Y - 0.74 - 100

= 0.3Y - 100

$$\therefore \lim_{y \to 500}(s) = \lim_{y \to 500}(s)(0.3Y - 100)$$

= 0.3 X 500 - 100

= 150 - 100

= 50

ii) S = Y - C_E

= Y - (150+0.84)

= Y - 0.8y - 150

$\lim\limits_{y \to 500} S$ = 0.2 (500) - 150

= 100 - 150

= -50

$\lim\limits_{y \to 500} C = \lim\limits_{y \to 500} [150 + 0.84]$

= 150+0.8 (500) = 150+400 = 550

Hence when income (Y) = 500 consumption (C) = 550 and saving (S) = - 50 i.e. there is negative saving of 50 (dissaving)

4.6. SUMMARY :

We end this lesson by summarizing what we have covered

1. The concept of sequence.
2. The limit of a function.
3. Continuous and discontinuous function.
4. Use of limit in economics.

4.7. LESSON END EXERCISE.

1) Write a note on a limit of a function.

2) Evaluate :-

(i) $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a}$

(ii) $\lim\limits_{h \to 0} \dfrac{(1+h)^n - 1}{h}$

(iii) $\lim\limits_{x \to 0} \dfrac{a^2 - 1}{b^x - 1}$

3) Give difference between limit and value of a function.

4.8. SELECTED READINGS

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PRINCIPLES OF DIFFERENTIATION, RULES OF DIFFERENTIATION, DIFFERENTIATION OF IMPLICIT FUNCTION, PARAMETRIC FUNCTION

STRUCTURE

5.1 INTRODUCTION

Differentiation is a mathematical technique of exceptional power and versatility. It is one of the two central concepts in the branch of mathematics called calculus and has a variety of applications in almost all branches of learning.

Differentiation which is mainly concerned with finding the rate of change in a given function y = f (x) with respect to the independent variable is very useful not only in Economics but in all other natural and social sciences. In Economics, we are frequently concerned with changes like growth, increasing and decreasing returns, costs etc. the concept of differentiation finds wide application in the field of marginal analysis, problems of optimization and the analysis of rate of change.

5.2 OBJECTIVES

After going through this lesson you should be able to understand

- Average, Marginal and instantaneous rates of change.
- Differentiation / derivative.
- Differentiation of standard functions.
- Rules / Law of Differentiation.
- Differentiation of implicit function, parametric functions.

5.3 AVERAGE RATE OF CHANGE AND INSTANTANEOUS RATE OF CHANGE

The average rate of change is the average value of the variable under consideration over an interval and instantaneous rate of change of a point is the limiting value of the average rate of change at that point.

Thus, if y = f(x) be a single value function of x, and if x changes to x+ Δx and y changes to y + Δy, the average rate of change in the function f(x) on the dependent variable y per unit.

Change in x is given by:-

$$\text{ARC} = \frac{Change\ \ in\ \ y}{Change\ \ in\ \ x} = \frac{\Delta y}{\Delta x} = \frac{(y+\Delta y)-y}{(x+\Delta x)-x}$$

$$\frac{f\,(x+\Delta x)-f\,(x)}{\Delta x} \quad or \quad \frac{f\,(x+h)\;-f\,(x)}{h}$$

If it is possible to calculate the limiting value of this expression $\Delta y/\ \Delta x$ as $\Delta\ x \rightarrow 0$, then the limiting value of the average rate of change is defined as the instantaneous rate of change (IRC) and is equal to

$$IRC = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x)}{\Delta x} = \lim_{n \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative (or Differential Co-efficient) and Differentiation:-

The derivative or differential co-efficient of a function say y = f(x) with respect to the independent variable x is the instantaneous rate of change or simply rate of change in the function f(x) w. r. t. x

Step I : Let y = f(x) (1)

Step II : If x changes to $(x+\Delta x)$ and y changes to $y + \Delta y$ so that the change (+uc/-ue increment) in x is Δ x and the corresponding change (or increment) in y is Δ y, then,

$$y + \Delta y = f\,(x + \Delta x) \quad (2)$$

Step III : Subtract (2) from (1) i.e. 2 - 1

$$y + \Delta y - y = f\,(x + \Delta x) - f(x)$$

$$\Delta y = f\,(x + \Delta x) - f(x) \quad (3)$$

Step IV : The ration of increment of y and x (also known as increment ratio differential and defined as the average rate of change in y or f(x) is obtained by dividing Δy with Δx is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(4)

Step V : Taking limits of both sides as $\Delta\ x \rightarrow 0$ we get:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad (5)$$

$\text{Lim} \frac{\Delta y}{\Delta x}$, defined as the instantaneous rate of change (IRC) in y or f (x), is written as $\frac{dy}{dx}$ *or* $\frac{d}{dx}$ (y) and is defined as the derivative of differential co-efficient by y w. r. t. (x). Thus,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (6)$$

The process of finding the derivative is called differentiation. Since five steps are involved this is known as derivative by five step rule or from first principle or from abinitio. The other notations for the differential co-efficient of y w. r. t. x are

$$\frac{dy}{dx}, \frac{df}{dx}, y', f'(x), y, f,$$
$$y_x, f_x : Dy, Df.$$

Graphical interpretation of the Derivative.

Let y = f (x) be a continuous function of x as shown by the curve AB fig. 1

Let p (x, y) be any point on the curve and let Q (x+ $\sqrt{x},$ y + $\sqrt{y}$) be another point in the neighborhood of P.

Join QP and extend the line to meet x-axis at R. Let $\angle$QR × Q. Draw PC and QD perpendiculars on OX and PE $\perp$ on DQ

Now PE = CD = OD - OC = δ x

EQ = δy

and $\dfrac{\delta y}{\delta x} = \dfrac{E\theta}{PE} = \tan\theta$

Now as $\theta \rightarrow$ P, $\delta x \rightarrow 0$ and the limit of $\delta y / \delta x$ If it exists, is called the derivative of y. w. r. t. x and is expressed as:-

$$\underset{\delta x \to 0}{\text{Lt.}} \frac{\delta y}{\delta x} = \frac{\text{Lt}}{\text{Qy}} \tan\theta = \tan\alpha$$

= Slope of the tan Pt.

= Slope of the curve at P.

= dy / dx.

Hence $\frac{dy}{dx}$ at any point P(x, y) on the curve f (x) is equal to the slope of the tangent at P (x, y). This is very important and useful result and is of much utility in economic theory.

5.3.1. Differentiation 'ab inito' from first principle

When derivatives are obtained without making use of established standard forms or theorems on differentiation, the technique of doing so, is described as the differentiation from first principle. The following steps i.e. involved.

i) Let y = f(x) be a function of one variable x.

ii) Let δx be an increment in the value of x and δy be the corresponding increment in the value of y.

$$\therefore\ y + \delta y = f(x + \delta x) \qquad (1)$$

iii) Find the increment in the dependent variable y, corresponding given increment in the dependent variable x, i.e.

$$\delta y = f(x+\delta x) - f(x) \qquad (2)$$

iv) Find the increment ratio by dividing both sides of 2 by

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} \qquad (3)$$

v) Proceeding to limits of $\delta y / \delta x$ as $\delta x \to 0$ gives the required derivative of the function f(x) w. r. t. x.

$$\frac{dy}{dx} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x) - f(x)}{\delta x}$$

The above expression may also be written as

$$\frac{dy}{dx} = \frac{Lt}{n \to 0} \frac{f(x+h) - f(x)}{h}$$

Example:- Find the derivative of x^2 w. r. t. x from first principle.

Sol :- $y = x^2$

Let δx be an increment in the value of x and δy the corresponding increment in y.

$$\therefore \quad y + \delta y = (\delta x + x)^2 = x^2 + 2x\delta x + (\delta x)^2$$

$$\delta y = 2x\ \delta x + (\delta x)^2$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

Proceeding to limits as $\delta x \to 0$, we get

$$\frac{dy}{dx} = \frac{Lt}{\delta x \to 0} (2x + \delta x) = 2x + 0 = 2x$$

Example:- Find, from definition, the derivatives of $\sqrt{x}$

Sol. Let y = $\sqrt{x}$

$$y + \delta y = \sqrt{x + \delta x}$$

$$\delta y = \sqrt{x + \delta x} - y$$

$$= \sqrt{x + \delta x} - y$$

$$= \sqrt{x + \delta x} - \sqrt{x} \times \frac{\sqrt{x + \delta x} + \sqrt{x}}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\delta y \quad \frac{x + \delta x - x}{\sqrt{x + \delta x} \quad \sqrt{x}}$$

$$= \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}}$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + \delta x + \sqrt{x}}}$$

Taking limits as $\delta x \to 0$, we get

$$\frac{dy}{dx} = \frac{Lt}{\sqrt{x} \to 0} \frac{1}{\sqrt{x + 5x} + \sqrt{x}} \frac{1}{\sqrt{x + 0 + \sqrt{x}}} = \frac{1}{2\sqrt{x}}$$

5.3.2 Derivatives of standard functions :-

The derivatives of standard functions will enable us to arrive at some standard forms which will help us to quicken the process of derivation.

Theorem:- Find the derivative of x^n w. r. t. x

Let $y = x^n$

$$\therefore\ y + \delta y = (x + \delta x)^n$$

$$\delta y = (x + \delta x)^n - \delta x$$

$$= x^n\ (1 + \frac{\delta x}{x})^n - x^n$$

$$= x^n\ \left\{(1 + \frac{\delta x}{x})^n \quad -1\right\}$$

Expanding by Binomial Theorem

$$\delta y = x^n \left[\left\{1 + n\frac{\delta x}{x} + \frac{n(n-1)}{2!}\right\}\left(\frac{\delta x}{x}\right)^2 + - - \right] - 1$$

$$= x^n \left[n.\frac{\delta x}{x} + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x!}\right)^2 - - \right]$$

Dividing both sides by δx, we get.

$$\frac{\delta y}{\delta x} = x^{n-1}\left[n + \frac{n(n-1)}{2!}\ \frac{\delta x}{x} + - - - \right]$$

Taking limits as $\delta x \rightarrow 0$, we get

$$\frac{dy}{dx} = x^{n-1}\ .\ n = nx^{n-1}$$

From this standard rule, in order to obtain the derivative of a power function such as x^n reduce the power of x by 1 and multiply it by original power. i.e.

$$\frac{d}{dx}\ (x^6) = 6\ x$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\ x^{-1/2}\ = -\frac{1}{2}\ x^{-3/2}$$

$$= -\frac{1}{2x^{3/2}}$$

Theorem : Obtain the derivative of $(ax+b)^n$ where n is any constant.

$$\text{Let } y = (ax+b)^n$$

$$y + \delta y = [a(x+\delta x) + b]^n$$

$$y + \delta y = [ax + b + a\delta x]^n$$

$$\delta y = [ax + b + a\delta x]^n - (ax+b)^n$$

$$= (ax+b)^n \left[1 + \frac{a\delta}{ax+b} - 1\right]$$

Because $\delta x \to 0,\ \frac{a\,\delta x}{ax+b} < 1,$ therefore $\left(1 + \frac{a\,\delta x}{ax+b}\right)$ can be expanded by binomial theorem.

$$\delta y = (ax+b)^n \left[\left\{1 + \frac{n\,a\,\delta x}{ax+b} + \frac{n(n-1)}{2!}\left(\frac{a\,\delta x}{ax+b}\right)^2 + ---\right\} - 1\right]$$

$$(ax+b)^n\ \frac{a}{ax}\ \frac{\delta x}{+b}\left[n + \frac{n(n-1)}{21}\ \frac{a\delta x}{ax+b} + --\right]$$

Dividing by δx, we get

$$\frac{\delta y}{\delta x} = (ax+b)^{n-1}\ a\left[n + \frac{n(n-1)}{2!}\ \frac{a\delta x}{ax+b} + --\right]$$

Proceeding to limits as $\delta x \to 0$, we get

$$\frac{dy}{dx} = (ax+b)^{n-1} a(n)$$

$$= na(ax+b)^{n-1}$$

Therefore the standard result is

$$\frac{dy}{dx} = \frac{d}{dx} \; (ax+b)^n \; = na(ax+b)^{n-1}$$

Rule: In order to obtain the derivative of any constant power of a linear function of the type (ax+b) as:-

Reduce the index by unity and multiply whole by original power and co-efficient of x.

The above rule can be used to write down the derivatives of function stated below at once.

$$\frac{d}{dx} \; (2x+12)^{4/5} \; = \frac{4}{5} \; (2x+12)^{\frac{4}{5}-1} \; 2$$

$$= \frac{8}{5} \; (2x+12)^{-1/5}$$

$$\frac{d}{dx} \; (4-2x)^6 \; = + (4-2x)^{6-1} \; (-2)$$

$$= -12(4-2x)^5$$

5.3.3 Basic theorems on differentiation

1. Derivative of a constant.

Prove that $\frac{d}{dx}\,(a) = 0$ for all x

Proof :- Let y = a where a is a constant

$$\therefore\ y + \delta y = a$$
$$\delta y = a - y$$
$$\delta y = a - a$$
$$\delta y = 0$$

Dividing both sides by δx, we have

$\dfrac{\delta y}{\delta x} = 0$ Proceeding to limits as $\delta x \to 0$, we get

$$\frac{dy}{dx} = 0$$
$$\frac{d}{dx}(a) = 0$$

Examples $\dfrac{d}{dx}(-6) = 0$

$\dfrac{d}{dx}(25) = 0$

2. Derivative of additive constant disappears on differentiation

Prove that $\dfrac{d}{dx}(u + a) = \dfrac{d}{dx}(u)$

Proof : Let $y = u + a$

$$y + \delta y = (U + \delta U) + a$$
$$= (U + a) + \delta U$$
$$\sqrt{y} = (U + a) + \delta U - (U + a)$$
$$\delta y = \delta U$$

Dividing both sides by δx

$$\frac{\delta y}{\delta x} = \frac{\delta}{\delta x}(U)$$

$$\text{Hence } \frac{d}{dx}(U+a) = \frac{d}{dx}(a)$$

Which shows that additive constant disappear on differentiation :-

Examples :-

$$\frac{d}{dx}(6x^2+5) = \frac{d}{dx}(6x^2) = 12x$$

$$\frac{d}{dx}(4x^3-6) = \frac{d}{dx}(4x^3) = 12x^2$$

Derivative of a multiplicative constant:-

Prove that $\frac{d}{dx}(aU) = a\frac{d}{dx}(U)$

Where a is a constant & U = f (x) is derivative at x.

Proof :- y = a U

$$y + \delta y = a(U+\delta U) = au + a\delta y$$

$$\delta y = aU + a\,\delta U - ay$$

$$\delta y = a\delta U$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = a\frac{\delta U}{\delta x}$$

Taking limits as $\delta x \to 0$, we get

$$\frac{dy}{dx} = a \frac{d}{dx} (U)$$

$$\text{Hence} \quad \frac{d}{dx} (Ua) = a \frac{d}{dx} (U)$$

$$\text{Example} \quad \frac{d}{dx} (6x^5) = \frac{d}{dx} (x^5)$$

$$= 6 \times 50 \ x^4$$

$$= 30 \ x^4$$

Derivative of sum or difference of Functions :-

Prove that :-

$$\frac{d}{dx} (u \pm v) = \frac{d}{dx} (u) \pm \frac{d}{dx} (v)$$

Where U & V are derivable functions at x.

Proof :- Let y = u ± v

$$\therefore \ y + \delta y = [(u + \delta U) + (v + \delta v)]$$

$$\delta y = \delta U + \delta v$$

Dividing by δx we get

$$\frac{\delta y}{\delta x} = \frac{\delta U}{\delta x} + \frac{\delta v}{\delta x}$$

Taking limits as $\delta x \to 0$, we get

$$\frac{dy}{dx} = \frac{d}{dx} (u) + \frac{d}{dx} (v)$$

Hence $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

Similarly we can prove

$$\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$$

Hence $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$

Therefore, the theorem states that the derivative of algebraic sum / difference of two function is equal to the corresponding algebraic sum of their derivatives, provided these derivatives and can be generalized to cover the case of more than two functions.

$$y = u_1 \pm u_2 + ... \pm u_n$$

$$\frac{dy}{dx} = \frac{du_1}{dx} + \frac{du_2}{dx} + \frac{du_3}{dx} + - + \frac{du_4}{dx}$$

Examples :-

1)
$$\frac{d}{dx}(6x^4 + 5x^3 - 4x^2 + 12)$$
$$= \frac{d}{dx}(6x^4) + \frac{d}{dx}(5x^2) - \frac{d}{dx}(4x^2) + \frac{d}{dx}(12)$$
$$= 6\frac{d}{dx}(x^4) + 5\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) + \frac{d}{dx}(12)$$
$$= 24x^3 + 15x^2 - 8x$$

2)
$$\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$
$$= \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(\sqrt{x})^{-1/2}$$
$$= \frac{d}{dx}(x)^{1/2} + \frac{d}{dx}(x)^{-3/2}$$
$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$$

Derivative of a product

Prove that $\frac{d}{dx}(uv) = u.\frac{d}{dx}(v)\ u\frac{d}{dx}(u)$ where u and v are two derivable functions of x

Proof :- Let y - uv

$$y + \delta y = (u+\delta U)(v+\delta v)$$
$$= uv - u\,\delta v + v\,\delta u + \delta u\,.\,\delta v$$
$$\delta y = u\ \delta v + v\ \delta u + \delta u\,.\,\delta v$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{u\,\delta v}{\delta x} + \frac{v\,\delta u}{\delta x} + \frac{\delta u\,.\,\delta v}{\delta x}$$

As $\delta x \to 0,\ \delta u \to 0.$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v.\frac{du}{dx}$$

Thus, the theorem states that the derivative of the product of two functions = first function × derivative of second + second function × derivative of first function:-

Example :- $\frac{d}{dx}(x+2)^2\ (x+5)^3$ Sol.

$$= (x+2)^2\ \frac{d}{dx}(x+5)^3 + (x+5)^3\ \frac{d}{dx}(x+2)^2$$
$$= (x+2)^2\ 3(x+5)^2\ (1) + (x+5)^3\ 2(x+2)1$$
$$= 3(x+2)^2(x+5)^2 + 2(x+5)^3(x+2)$$
$$=(x+2)\ (x+5)^2\ [3(x+2) + 2(x+5)]$$
$$= (x+2)\ (x+5)^2\ [3x+6+2x+10]$$
$$= (x+2)(x+5)^2(5x+16)$$

Example :- If $(x + 1/x)\,(\sqrt{x} + 1/\sqrt{x})$

Sol. $\frac{dy}{dx} = \left(x+\frac{1}{x}\right)\frac{d}{x}\left(\sqrt{x} + 1/\sqrt{x}\right) + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\frac{d}{dx}\left(x+\frac{1}{x}\right)$

$$= \left(x+\frac{1}{x}\right)\frac{d}{dx}\left[(x)^{1/2} + x^{-1/2}\right] + \sqrt{x} + \frac{1}{\sqrt{x}}\left(1-\frac{1}{x^2}\right)$$

$$= \left(x+\frac{1}{x}\right)\left[\frac{1}{2}(x) - \frac{1}{2}x^{-3/2}\right] + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(1-\frac{1}{x^2}\right)$$

$$= \left(x+\frac{1}{x}\right)\left[\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right] + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(1-\frac{1}{x^2}\right)$$

Simply

$$= \frac{x-1}{2x^{5/2}}\,(3x^2 + 4x + 3)$$

Derivative of a Quotient of two functions

Prove that :- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Where u and v are both derivable functions at x and $v \neq 0$.

Proof :-

$$y = \frac{u}{v}$$

$$y + \delta y = \frac{u + \delta U}{v + \delta v}$$

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$= \frac{v\,(u + \delta u - u\,(v + \delta v}{(v + \delta v)\ v}$$

$$= \frac{v\,\delta u - u\,\delta v}{(v + \delta v)\,v}$$

Dividing both sides by δx we get:-

$$\frac{\delta y}{\delta x} = \frac{v\ \delta u/\delta x\ - u\ \delta v/\delta x}{v\ (v+\delta v)}$$

Proceeding to limits as $\delta x \to 0\ \&\ \delta v \to 0,$ we get

$$\frac{dy}{dx}\left(\frac{u}{v}\right) = v.\ \frac{du/dx - v.\ dv/dx}{u^2}$$

Hence

$$\frac{d}{dx}\left(\frac{u}{v}\right) = v.\ \frac{du/dx - v.\ dv/dx}{u^2}$$

The rule:-

$$\frac{d}{dx}\ \text{(Quotient of the two functions)}$$

$$= \frac{\text{Denominator} \times \frac{d}{dx}\ (\text{Numerator}) - \text{Num}\ \frac{d}{d_x}(\text{Denom .})}{(\text{Denom .})^2}$$

Examples:-

1)
$$\frac{d}{dx}\left(\frac{5x^2+1}{x}\right) = \frac{x\ \frac{d}{d_2}(5x^2+1) - (5x^2+1)\frac{d}{dx}(x)}{x^2}$$

$$= x.\ \frac{(10\,x) - \left((5x^2+1(1)\right)}{x^2}$$

$$= \frac{10\,x^2 - 5x^2 - 1}{x^2} = \frac{5x^2 - 1}{x^2}$$

2) $$\frac{d}{dx}\left(\frac{3x}{7x^2+8}\right)\ \text{w.r.t.}\ \ x$$

$$\frac{d}{dx}\left(\frac{3x}{7x^2+8}\right) = \frac{(7x^2+8)\frac{d}{dx}(3x) - 3x\,\frac{d}{dx}(7x^2+8)}{(7x^2+8)^2}$$

$$= \frac{(7x^2+8)\,3 - 3x\,(14x)}{7x^2+8}$$

$$= \frac{21x^2 + 24 - 42x^2}{7x^2+8}$$

$$= \frac{24 - 21x^2}{7x^2+8}$$

3) $\frac{d}{dx}\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)$

$$= \frac{(1-x)^{1/2}\,\frac{d}{dx}(1+x)^{1/2} - (1+x)^{1/2}\,\frac{d}{dx}(1-x)^{1/2}}{(\sqrt{1-x})^2}$$

$$= \frac{(1-x)^{1/2}\,\frac{1}{2}(1+d)^{-1/2}\,(1) - (1+x)^{1/2}\left(\frac{1}{2}\right)(1-x)^{-1/2}\,(-1)}{(\sqrt{1-x})^2}$$

$$= \frac{1}{2}\,\frac{(1-x)^{1/2}}{(1+x)^{1/2}} + \frac{(1+x)^{1/2}}{2(1-x)^{1/2}}$$

$$= \frac{1}{2}\left[\frac{(\sqrt{1-x})\,(\sqrt{1+x}) + (\sqrt{1+x})\,(\sqrt{1-x}}{1-x}\right]$$

$$= \frac{2\,(1+x)^{1/2}\,(1-x)^{1/2}}{1-x}$$

$$= \frac{1}{(1+x)^{1/2}\,(1-x)^{1/2}} \times \frac{1}{1-x}$$

$$= \frac{1}{(1+x)^{1/2}\,(1-x)^{3/2}}$$

Differentiation of a function chain rule if y is a function u and is a function of x then:-

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Let δx be an increment in x and δu be the corresponding increment in u and the corresponding in y be δy

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

Proceedings to limits as $\delta x \to 0$, δu & $\delta y \to 0$, we get

$$\delta x \to 0 \qquad \delta u \to 0 \qquad \delta x \to 0$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{dy}{dx}$$

$$\text{Lim}\frac{\delta y}{\delta x} = \text{Lim}\frac{\delta y}{\delta u} \cdot \text{Lim}\frac{\delta u}{\delta x}$$

Examples:- If $y = 2w^2 + 1$, $w = 2z^2$ $Z = 2x + 3x^2$, find the derivative of y w. r. t. x

We know $\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dz} \cdot \frac{dz}{dx}$

$$y = 2w^2 + 1, \therefore \frac{dy}{dw} = 4w$$

$$w = 2z^2 \quad \frac{dw}{dz} = 4z$$

$$z = 2x + 3x^2 \quad \frac{dz}{dx} = 2 + 6x$$

$$\therefore \frac{dy}{dx} = \left(\frac{dy}{dw}\right)\left(\frac{dw}{dz}\right)\left(\frac{dz}{dx}\right)$$

$$= (4w)(4z)(2 + 6x)$$

As But, $w = 2z^2$, $z = 2x + 3x^2$

$$= 4(2z^2)\, 4(2x + 3x^2)(2 + 6x)$$

$$= 32(2x + 3x^2)^2 (2x + 3x^2)(2 + 6x)$$

$$= 32(2x + 3x^2)(2 + 6x)$$

Example:- $y = \sqrt{2x^3+4}$

$\sqrt{u}$ Where $u = 2x^3 + 4$

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{u} = \frac{1}{2u^{1/2}}$$

$$= \frac{1}{2(2x^3+4)^{1/2}}$$

$$\frac{du}{dx} = 6x^2$$

$$\therefore \quad \frac{dy}{dx} = (dy/du)\left(\frac{du}{dx}\right)$$

$$= \frac{6x^2}{2\sqrt{2x^3+4}}$$

Let y = $$= \frac{3x^2}{2x^3+4}$$

5.3.4 Differentiation of implicit function

Consider y = $\dfrac{2x}{4x^2+5}$

This is an explicit function because it express y directly in terms of x. When this is the form $4x^2y$ - 2x+5, it becomes implicit function and y is said to be defined implicit as a function of x.

Example :- Find $\dfrac{dy}{dx}$ when $x^2 + y^2 + 2y = 20$

Sol. We have

$$x^2 + y^2 + 2y = 20$$

Differentiation both sides w. r. t. x considering y as a function of x

$$2x + 3y\frac{dy}{dx} + 2\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$(1 - 4)\ dy/dx = -x$$

$$dy / dx = - \frac{x}{1+y}$$

Note:- If we consider x as a function y, then $\frac{dy}{dx}$ can be obtained by diff. the function w. r. t. y thus.

$$2x \frac{dx}{dy} + 2y + 2 = 0$$

$$x \frac{dx}{dy} + y + 1 = 0$$

$$x \frac{dx}{dy} = - (y + 1)$$

$$\frac{dx}{dy} = - \frac{y+1}{x}$$

Example :- If $x^2 + 5x^2 y + yx = 5$. Find dy /dx

Sol :- Given $x^3 + 5x^2 y + yx = 5$

Differentiating both sides w. r. t. x, we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(5x^2 y) + \frac{d}{dx}(yx) = \frac{d}{dx}$$

$$3x^2 + \left[10xy + 5x^2 \frac{dy}{dx}\right] + \left[x . \frac{dy}{dx} + y\right] = 0$$

$$5x^2 \frac{dy}{dx} + x . \frac{dy}{dx} = - 3x^2 - 10xy - y$$

$$\frac{dy}{dx} = - \frac{3x^2 - 10xy - y}{5x^2 + x}$$

Example: Find $\frac{dy}{dx}$ when $x^{2/3} + y^{2/3} = a^{2/3}$

Sol :- $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating both sides w. r. t. x

Consider y as a function of x

$$\frac{2}{3} x^{2/3-1} + \frac{2}{3} y^{2/3-1} \frac{dy}{dx} = 0$$

$$\frac{2}{3} y^{-1/3} \frac{dy}{dx} = -\frac{2}{3} x^{-1/3}$$

$$\frac{dy}{dx} = -\frac{d}{y^{-1/3}}$$

$$= -\frac{y}{x^{1/3}}$$

Example : If $ax^2 + 2hxy + b y^2 + 2gx + 2fy + c = 0$

Find dy / dx

Sol :- $ax^2 + 2hxy + b y^2 + 2gx + 2fy + c = 0$

$$a \frac{d}{dx}(x^2) + 2h \frac{d}{dx}(xy) + b \frac{d}{dx}(y^2) + 2y \frac{d}{dx}(x)$$

$$+ 2f \frac{d}{dx}(y) + \frac{d}{dx}(1) = 0$$

$$a\ 2x + 2h\left[y + x\frac{dy}{dx}\right] + b\ 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$(2hx + 2by + 2f)\frac{dy}{dx} = -2ax - 2hy - 2g$$

$$\therefore \frac{dy}{dx} = -\frac{(ax + hy + g)}{hx + by + f}$$

Example : Find $\frac{dy}{dx}$, when y = $\sqrt{x + \sqrt{x + \sqrt{x} + ... \infty}}$

Sol :-

$$y = \sqrt{x + \sqrt{x} + \sqrt{x} + ... \infty}$$

$$y = \sqrt{x + y}$$

$$y^2 = x + y$$

Differentiating both sides w. r. t. x

$$2y \frac{dy}{dx} \pm 1 + \frac{dy}{dx}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$(2y - 1) = 1 \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{2y-1}$$

5.3.5 Parameter Equations and their derivatives :-

If x and y are both functions of a third variable, say t, then the equations are called parametric equations. The variable t is called parameter.

x = f (t) (1)

y = f (t) (2)

The equation on 1 & 2 are known as parametric equations

Example : If $x = 5t^2$ and $y = 3t^2$, find dy / dx

Sol :- $x = 5t^2$ $y = 3t^2$

$$\frac{dd}{dt} = 10\,t \quad \frac{dy}{dt} = 6\,t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy\ /\ dt}{dx\ /\ dt} = \frac{6\,t}{10\ t} = \frac{3}{5}$$

Example : Find $\frac{dy}{dx}$ when $x = \frac{3t}{1+t^3}$, $\frac{3at^2}{1+t^3}$

Sol : $x = \frac{3at}{1+t^3}$

$$\therefore \quad \frac{dx}{dt} = \frac{(1+t^3)\ 3a - 3at\ \frac{d}{dt}\ (3\ 1+t^3)}{(1+t^3)}$$

$$= \frac{(1+t^3)\ 3a - 3at\ (3t^2)}{(1+t^2)}$$

$$= \frac{3a\ + 3at^3 - 9\,at^3}{(1+t^3)^2}$$

$$= \frac{3a - 6at^3}{(1+t^3)^2} = \frac{3a\ (1-2t^3)}{(1+t^3)^2}$$

$$y = \frac{3at^2}{1+t^3}$$

$$\frac{dy}{dt} = \frac{(1+t^3)\frac{d}{dt}3at^2 - 3at^2\frac{d}{dt}(1+t^3)}{(1+t^3)^2}$$

$$= \frac{(1+t^3)\ 3at - 3at^2\ 3t^2}{(1+t^3)^2}$$

$$= \frac{6at + + at^4 - 9at^4}{(1+t^3)^2}$$

$$= \frac{3at\ (2-3t^3)}{(1+t^3)^2}$$

$$\frac{dy}{dx} = \frac{dy\ /\ dt}{dx\ /\ dt} = \frac{3at\ \ (2-t^3)}{(1+t^3)^2} \times \frac{(1+t^3)^2}{3at\ \ (1-2t^3)}$$

$$= \frac{2t - t^4}{1-2t^3}$$

$$= \frac{t(2-t^3)}{1-2t^3}$$

5.4 SUMMARY

We conclude this lesson by summarizing what we have covered in it.

	FUNCTION	DERIVATIVE
i)	$\frac{d}{dx}(x^n)$	nx^{n-1}
ii)	$\frac{d}{dx}(ax+b)^n$	$na(ax+b)^{n-1}$
iii)	$\frac{d}{dx}(U+a)$	du/dx

iv) $\frac{d}{dx}$ (an) a. du/dx

v) $\frac{d}{dx}(u \pm v)$ $\frac{d}{dx}(U) + \frac{d}{dx}(v)$

vi) $\frac{d}{dx}(Uv) \quad U\frac{d}{dx}(v) + v\frac{d}{dx}(U)$ Derivative of two function (Product is first function × derivative second + second function × derivative of first function.

vii) $$\frac{d}{dx}(U/V) = \frac{v\frac{d}{dx}(U) - U\frac{d}{dx}(v)}{v^2}$$

i.e. Derivative of two functions of the type U/V is

$$\frac{D^r \frac{d}{dx}(N^r) - N\frac{d}{dx}(D^r)}{Dr^2}$$

viii) Chain rule : If y is a function of U and U is a function x, then

$$\frac{dy}{dx} \; \frac{dy}{du} \times \frac{du}{dx}$$

ix) Differentiation of implicit function.

x) Differentiation of parametric function.

5.5 LESSON END EXERCISE

Q.1. a) Find the differential co-efficient of $(ax+b)^n$, from first principle.

b) Obtain, the derivative of x^n w. r. t. x

Q.2. If $y = \left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$, find the derivative of y w. r. t. x.

Q.3. If $y = \frac{(x+2)(2x+1)}{x^3 - 1}$, find dy / dx

Q.4. If $y = 2w^2 + 1$, $w = 2$, $z^2 = 2x + 3x^2$, find the derivative of y w. r. t. x.

Q.5. Find dy / dx

(i) $x^5 + y^5 + 5xy - c = 0$, ii) $x^3 + y^3 = 3xy$

5.6 SUGGESTED READINGS

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PARTIAL AND TOTAL DIFFERENTIATION EXPANSION BY TAYLOR SERIES, ALLIED ECONOMIC APPLICATIONS

STRUCTURE

6.1 INTRODUCTION

So far we have considered functions of only one independent variables viz., $y = f(x)$. But in most of the problems in economics we are frequently confronted with functions of more than one (independent) variable. For example.

i) Quantity demanded (x) of a commodity depends not only on the price (p) of the commodity but on the prices (p_1, p_2, ….) of other commodities, income (y) of the consumer etc. i.e. x is a function of more than one (independent) variable e.g., $x = x^d = f(p_1, p_2, \ldots, y)$.

ii) Quantity supplied (x) of a commodity is a function of not only price (P) of the commodity but a function of the prices (p^1, p^2, ….) of the technology (T) etc. i.e., $x = xs = (p_1, p_2, \ldots, p_1, p_2, \ldots T)$.

iii) The utility (u) derived by the consumer depends on the quantity consumed of various commodities (x_1, x_2 …….) i.e. $U = f(x_1, x_2 \ldots\ldots)$.

iv) The level of production (or output) Q of a commodity depends on the inputs (L&K) etc. i.e. $Q = f(L, K)$ etc.

6.2 OBJECTIVES

After reading this lesson, you would be able to understand.

- Function of several variables.
- The partial derivates with examples.
- Total differential, total derivative and examples.
- Application.

6.3 FUNCTION OF TWO VARIABLES

Let U be a symbol which has one definite value for every permissible pair of values of the independent variables x and y then u is called a function of two variables x & y, we write it as :-

$$U = f(x, y) \text{ or } u(x, y)$$

Similarly, a function of n variables may be written as:-

$U = U(x_1, x_2 \ldots\ldots x_n)$ or $U = f(x_1, x_2 \ldots\ldots x_n)$

Where $x_1, x_2, \ldots\ldots x_n$ are the n independent variables and corresponding to each set of values of the n independent variables $(x_1, x_2 \ldots\ldots x_n)$ we get on definite value of u.

Value of a function of two and more variables

If $U = f(x, y)$ is a function of two variables, then $f(x_1, y_1)$ is known as the value of function at the point (x_1, y_1) i.e. when $x = x_1$ and $y = y_1$

For example :-

If $U = x^2 - y^2 + 2xy - 10$ then

i) $f(0, 0)$ = value of the function at $x = 0, y = 0 = 0 + 0 + 2(0)(0) - 10$

$= -10.$

ii) $f(-5, 2) = (-5)^2 - (2)^2 + 2(-5)(2) - 10 = 25-4-20-10 = -9$

iii) $f(2,-1) = (2)^2 - (-1)^2 + 2(2)(-1) - 10 = 4 - 1 - 4 - 10 = 11.$

If $U = f(x_1, x_2, x_3)$ is a function of three variables, then $f(x_{11}, x_{21}, x_{31})$ is called value of the function at the point (x_{11}, x_{21}, x_{31}).

For Examples if $U = x_1^2 + x_2^2 + x_3^3$ then,

$f(0, 0, 0) = (0)^2 + (0)^2 + (0) = 0$

$f(1, 2, 3) = (1)^2 + (2)^2 + (3)^2 = 1+4+9 = 14$

$f(0, 2, 0) = (0)^2 + (2)^2 + (0)^2 = 4.$

Functions of two and more variables i.e. multi-variable functions in economic theory

(i) **Demand Function :-** $x^d_i = f_I(p_1, p_2 ; v)$ [i = 1, 2]

a) If there are two consumer goods X_1 and X_2 with x_1, & x_2 as the quantity demanded of the goods X_1 and X_2 and p_1, p_2 the prices of the two goods respectively and y is the income of the consumer, then

$$x_1^d = f_1(p_1, p_2 ; y)$$

$x_2^d = f_2 (p_1, p_2 ; y)$

b) If there are n consumer good $x_1, x_2 \ldots, x_n$ with prices $p_1, p_2 \ldots p_n$ and quantity demanded of the n goods as $x_1, x_2, \ldots x_n$ respectively and y is income of the consumers.

$x_1^d = f_1 (p_1, p_2 \ldots\ldots p_n ; y)$

$x_2^d = f_2 (p_1, p_2 \ldots\ldots p_n; y)$

ii) **The supply function:-**

$x^s_1 = f_1 (p_1, p_2 ; p_1, p_2 ; T)$

If there are two consumer goods X_1, X_2 with quantity demanded and prices of the two goods as x_1, x_2 & p_1, p_2 respectively. p_1, p_2 being prices of 2 inputs and T being the level of technology, then.

$X_1^s = f_1 (p_1, p_2 ; p_1, p_2 ; T)$

$X_2^s = f_2 (p_1, p_2 ; p_1, p_2 ; T)$

iii) **The utility function:-** $U = U(X_1, X_2)$ if there are 2 commodities X_1 and X_2 with x_1, x_2 as the quantities consumed respectively of the 2 commodities, then the level of satisfaction derived by a consumer i.e. the utility function of the consumer can be written as:-

$U = f (x_1, x_2)$

In case of n commodities

$U = f(x_1, x_2 \ldots\ldots X_s)$ for n goods

iv) **The production function:-** $Q = f (x_1, x_2)$ if x_1, x_2 are the quantities of two inputs and Q is the level of output, then the production function in terms of inputs can be written as:-

$Q = f (x_1, x_2)$

$Q = f(x_1, x_2, \ldots\ldots x_n)$ for n inputs

v) **The Cost Function :-** $C = f(x_1, x_2)$ if a firm produces 2 commodities X_1 & X_2 with x_1, x_2 as the levels of output of the 2 commodities then the cost function C can be written as function of level of output $x_1 + x_2$.

$$C = f(x_1, x_2)$$

$$C = f(x_1, x_2 \ldots\ldots x_n) \text{ for n commodities}$$

Self-Assessment - I

1. What do you mean by Supply function and the demand function.

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2. Explain the meaning of utility function and the production function.

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6.4 THE PARTIAL DERIVATIVES :-

Definition:- The partial derivatives of U = f (x, y) with respect to x at the point (x, y) is defined as

$$\frac{Lt}{\delta \to 0} \quad \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

If it exists finitely, Thus, while finding partial derivatives of U= f (x, y) with respect to x at (x, y), we assume that y remains fixed and the change in the function is due to change in x (from x to x+ δx)

The partial derivative of U = f(x, y) w. r. t. x at (x, y) is denoted by $\frac{\partial U}{\partial x}$ f(x) of fx.

Similarly, the partial derivative of U = f (x, y) w. r. t. y at (x, y) is defined as:-

$$\underset{\delta y \to 0}{Lt} \quad \frac{f(x, y + \delta y) - f(x, y)}{\delta y},$$

If it exists finitely and is, defined by

$$\frac{\partial U}{\partial y} \ \ or \ \ f_y \ \ (x, y) \ \ or \ \ fy$$

Remarks :- (i) The concept of partial derivatives is applicable to functions of more than one variable, i.e. function of two or more variables.

6.4.1 Second order partial derivatives

The partial derivatives of the first order partial derivatives are known as the second order partial derivatives e. g.

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\right] \ or \ \frac{\partial^2 f}{\partial x^2} \ \ or \ \ U_{xx} \ or \ f_{xx} \ or \ f_{11}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial y}\right] \ or \ \frac{\partial^2 f}{\partial y^2} \ \ or \ U_{yy} \ or \ f_{yy} \ \ or \ f_{22}$$

$$\frac{\partial^2 U}{\partial_y \ \partial_x} = \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial Y}\right] \ or \ \frac{\partial^2 f}{\partial_y \partial_x} \ \ or \ \ f_{xy} \ \ or \ \ f_{12} \ \ or \ U_{12}$$

$$\frac{\partial^2 U}{\partial_x \ \partial_y} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\right] \ or \ \frac{\partial^2 f}{d_y \ d_x} \ \ or \ f_{yx} \ \ or \ \ f_{21} \ \ or \ \ U_{21}$$

Examples :- Find the first order and second order partial derivatives, $3x^2 y^2 + x^5 + 3y^2$.

Solution :-

$U = 3x^2 y^2 + x^5 + 3y^2$

$f(x) = \dfrac{\partial U}{\partial x} = 6xy^2 + 5x^4$

$$f(y) = \frac{\partial U}{\partial x} = 6x^2y + 6y$$

$$f_{xx} = \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\right]$$

$$= \frac{\partial}{\partial x}\left[6xy^2 + 5x^2\right]$$

$$= 6y^2 + 20x^3$$

$$f_{yy} = \frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial y}\right]$$

$$= \frac{\partial}{\partial y}\left[6x^2 y + 6y\right]$$

$$= 6x^2 + 6$$

$$f_{xy} = \frac{\partial^2 U}{\partial y.\partial x} = \frac{\partial}{\partial y}\left[\frac{\partial U}{\partial x}\right]$$

$$= \frac{\partial}{\partial y}\left[6xy^2 + 5x^4\right]$$

$$= 12xy$$

$$f_{yx} = \frac{\partial^2 U}{\partial_x . \partial_y} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[6x^2 y + 6y\right]$$

$$= 12xy$$

From above we find

$f_{xy} = f_{yx}$

Example : Find the first order partial derivative

$$U = \frac{2x^2}{x + y + 8}$$

Sol.

$$\frac{\partial U}{\partial x} = \frac{(x+y+8)\,\frac{\partial}{\partial x}(2x^2) - 2x^2\,\frac{\partial}{\partial x}\,(x+y+8)}{(x+y+8)^2}$$

$$\frac{\partial U}{\partial x} = \frac{(x+y+8)(4x) - 2x^2\ (1)}{(x + y + 8)^2}$$

$$= \frac{4x^2 + 4xy + 32x - 2x^2}{(x+y+8)^2}$$

$$= \frac{2x\ (x + 2y + 16)}{(x+y+8)^2}$$

$$\frac{\partial U}{\partial y} = 2x^2\ \frac{\partial}{\partial y}\ (x+y+8)^{-1}$$

$$= 2x^2\ (-1)\,(x+y+8)^{-2}\ (1)$$

$$= -\ \frac{2x^2}{(x+y+8)^2}$$

Self-Assessment - II

1. Find the second order partial derivative of $e^{x2} - y^{2}$.

2. Show that :

$$\frac{\partial u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\ (a^2 + b^2) - 4$$

6.4.2. Signs of Partial Derivatives:-

A function of two variables yields two first order partial derivatives, f_x shows the rate of change of the function U = f(x,y) w. r. t. x, assuming y as a constant and f_y, shows the rate of change of the function.

U = f (x, y) w. r. t. y assuming x as a constant f_{xx} shows whether the function is increasing or decreasing or constant rate, when x varies and y remains constant.

1) $f_x > 0$ means that the function increases as x increase, y remaining constant ; $f_x<0$ means that the function decreases as x increases y being held constant.

2) $f_{xx}>0$ means that the rate of change of the function increases as x increases y being held constant, $f_{xx}<0$ means that the function changes at decreasing rate. Similarly we can interpret the signs of f_y & f_{yy}.

3) $f_{xy} = f_{yx}<0$ means both that f_x decreases as y increases & f_y decreases as x increases.

4) $f_{xy} = f_{yx} >0$, means that f_x increases as y increases and f_y increases as x increases.

5) $f_{xy} = f_{yx} = 0$ means that there is no interaction between the variables.

Example :- A demand function is given by

$$Q = 5y + 4y^3 - 10p^2 - 80p^{-5}, \; y>0, \; p>0$$

Where q is the quantity demanded, y is income and p is the price

a) Find the slope of demand curve.

b) Is the commodity normal or inferior.

c) Is the reaction of demand to price independent of the level of income.

Solution :- We have

$$q = 5y + 4y^2 - 10\,p^2 - 80\,p^{-5}$$

$$\frac{\partial q}{\partial p} = fp = -30\,p^2 - 400\,p^{-6}$$

$\therefore$ Slope of demand curve is

$fp = -30p^2 - 400\ p^{-6}$

b) If the rate of change of q w. r. t. t increases as y increases, the commodity will be normal, thus.

$$\frac{\partial q}{\partial y} = fy = 5 + 12y^2$$

Since y is positive, therefore $f_y > 0$. Hence the commodity is a normal good.

c) $f_{py} = f_{yp} = 0$

When cross partial derivatives are zero, then the reaction demanded to price is independent of the level of income i.e. it neither increases nor decreases as y increases. Also f_y neither increases not decrease thus, the rate of change of q w. r.t. price is not independent of income level. Derivative of f_p w.r.t.y is zero and derivative of f_y w. r. t. p is also zero. At any particular level of price, the responsiveness of demand to price does not depend upon income level. The level of income will be high or low, the value of f_p will remain the same. In such cases the value of the function will be the sum of separate effects of the independent variables. Hence our answer to (c) part is yes.

Example:- For the Cobb-Douglas production function U = A $x^{\alpha} \cdot y^{1-\alpha}$ find the marginal products of labour and capital where x is labour and y is capital. Also show that

$$x\ \frac{\partial U}{\partial x} + y.\ \frac{\partial U}{\partial y} = U$$

Solution : $U = A\ x^{\alpha}\ \ y^{1-\alpha}$

MP_L = Marginal product of labour

$$\frac{\partial}{\partial x}(U) = \frac{\partial}{\partial x}(U) = \frac{\partial}{\partial x}\left[Ax^{\alpha}\ y^{1-\alpha}\right]$$

$$= A.\ y^{1-\alpha}\ \ x^{\alpha}.\ \ x^{\alpha-1}$$

$$= A\alpha. \ \frac{x^{\alpha}}{x} \ . \quad y^{1-\alpha}$$

$$= \frac{\alpha}{x} \ \left[A. \ x^{\alpha} \ \ y^{1-\alpha}\right]$$

$$= \frac{\alpha}{x} \ U$$

MP_k = Marginal product of capital

$$\frac{\partial}{\partial xy} \ (U) \ \ \frac{\partial}{\partial y} \left[A. \ \ x^{\alpha}. \ v^{1-\alpha}\right]$$

$$= \ A. \ x^{\alpha} \ \ (1-\alpha) \, y^{-\alpha}$$

$$= \frac{A(1-\alpha)}{} \quad x \ \ \frac{y^{1-\alpha}}{y}$$

$$= \frac{1-\alpha}{y} \quad \left[A. \ x^{\alpha} \ \ y^{1-\alpha}\right]$$

$$= \frac{(1-\alpha) \ U}{y}$$

Hence, $x\dfrac{\partial U}{\partial x} + y \, \dfrac{\partial U}{\partial x} = x\left[\dfrac{\alpha u}{x}\right] + y\left[\dfrac{(1-\alpha)\,U}{y}\right]$

$$= \alpha \ u + (1-\alpha) \, u$$

$$= \ u \ (\alpha + 1 - \alpha)$$

$$= u = R. \ H. \ S.$$

6.5 TOTAL DIFFERENTIAL AND TOTAL DERIVATIVE:-

Definition:- Total differential : If U = f (x , y) is a function of two independent variables, x and y, then total differential of U to be denoted by du, is a linear approximation of the change in u or f (x, y) when there is a small change in both x and y and we write it as:-

$$\Delta u = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = f_x \; dx + f_y . dy$$

We shall explain this concept with the help of an example of wheat output (u), land x and labour y, our problem is to find out as to what will be the change in wheat output (u) when there is a small change in both land (x) and labour (y). We know that $\frac{\partial U}{\partial x}$ denotes the change in (u) due to small unit change in x and keeping y constant. Thus if land x changes by Δx, keeping y constant, change in u due to a small unit.

Change in x = $\left(\frac{\partial U}{\partial x}\right)$ (Δx) (1)

Similarly, if there is a change of Δy in land y, when the change in u due to a change of Δy in y keeping x constant.

= (Change in U due to small unit change in y) × (change in y)

$$= \left(\frac{\partial U}{\partial x}\right) (\Delta y) \qquad (2)$$

$\therefore$ as a first approximation, we can think that the change in U due to a small change of Δx in x and a small change of Δy in y will be

$$\Delta u = \left[\frac{\partial U}{\partial x} \quad \Delta x\right] + \left[\frac{\partial U}{\partial y} \quad . \quad \Delta y\right]$$

= fx ∂ x + fy. ∂ y.

i.e. Total change in U [due to small change in x & y] = [Change in U due to (total) change in x (keeping y constant) & change in U due to change in y (Keeping x constant) i.e. ∂U = [Change in U due to a unit of change in x) (total change in x) + (change in U due to a unit change in y) (total change in y).

$$du = \frac{\partial U}{\partial x} dx + \frac{\partial u}{\partial y} dy = f_x \; dx + f_y \; d_y$$

6.6 DEFINITION:- TOTAL DERIVATIVE

a) Let U = f(x, y) be function of two independent variables in x and y, then total change in u w. r. t. one of the variables (say x) when there is a change in both x and y, is called the total derivative of u w. r. t. x and is denoted by $\frac{du}{dx}$, similarly, the total change u w. r. t. y when there is change in both x and y is called the total derivative of u w. r. t. y and is written as $\frac{du}{dy}$

$$Thus, \ \frac{dU}{dx} = \left(\frac{\partial U}{\partial x}\right)\left(\frac{dx}{dx}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{du}{dy}\right) = \frac{\partial U}{\partial x} + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dx}\right)$$

$$= f_x + f_y \frac{dy}{dx}$$

$$\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial y}\right)\left(\frac{dx}{dy}\right) + \left(\frac{\partial u}{\partial y}\right)\frac{du}{dy} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dy} + \left(\frac{\partial u}{\partial y}\right)$$

$$= f_x \left(\frac{dx}{dy}\right) + f_y$$

6.6.1 Illustrative examples of total differential

i) $U = x^3 + 3x^2 y + 6xy^2 + 2y^3$

Total differential $\partial U = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = f_x \, dx + f_y \, dy$

$$\frac{\partial y}{\partial x} = 3x^2 + 6xy + 6y^2 = 3x^2 + 6xy + 6y^2$$

$$\frac{\partial U}{\partial y} = 0 + 3x^2 + 12xy + 6y^2 = 3x^2 + 12xy + 6y^2$$

$$\therefore \ du = (3x^2 + 6xy + 6y^2)\, dx + (3x^2 + 12xy + 6y^2)\, dy$$

ii) $U = xy^3 - yx^3$

$$\frac{\partial u}{\partial x} = y^3 - 3yx^2$$

$$\frac{\partial u}{\partial y} = 3xy^2 - x^3$$

$$du = (y^3 - 3x^2 y)\,dx + (3y^2 x - x^3)\,dy$$

iii) $$U = \frac{x^2 - y^2}{x^2 - y^2}$$

$$\frac{\partial U}{\partial x} = \frac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2x}{(x^2 + y^2)^2}\left[x^2 + y^2 - x^2 + y^2\right]$$

$$= \frac{2x}{(x^2 + y^2)^2}\quad 2y^2 \quad = \frac{4xy^2}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2 + y^2)(-2y) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2}$$

$$= \frac{-2y}{(x^2 + y^2)^2}\left[x^2 + y^2 + x^2 - y^2\right]$$

$$= \frac{-2y\,(2x^2)}{(x^2 + y^2)^2} = \frac{-4yx^2}{(x^2 + y^2)^2}$$

$$\therefore\ du = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy$$

$$= \frac{4xy^2}{(x^2 + y^2)^2}\,dx + \frac{(-4yx^2)}{(x^2 + y^2)^2}\,dy$$

$$= \frac{4xy}{(x^2 + y^2)^2} \left[y\,dx - x.\,dy\right]$$

iv) Find d if $q = f(L, K) = A.\,L^{3/4}\ K^{3/4}$

Sol.

$$q = A\ L^{3/4}\ \ K^{3/4}$$

$$\frac{\partial q}{\partial L} = \frac{3}{4}\ A.\ L^{-1/4}\ \ K^{3/4} = \frac{3}{4}\,A\,K^{3/4}\ \ L^{-1/4}$$

$$\frac{\partial q}{\partial k} = \frac{3}{4}\ A.\ L^{3/4}\ K^{-1/4} = \frac{3}{4}\,A\,K^{3/4}\ \ L^{-1/4}$$

$$\frac{dq}{\quad} = \frac{\partial q}{\partial L}\ dL + \frac{\partial q}{\partial k}\ dk$$

$$= \left(\frac{3}{4}\ A.\,K^{3/4}\ \ L^{-1/4}\right) dL + \left(\frac{3}{4}\ AL^{3/4}\ \ K^{-1/4}\right) dK$$

$$= \frac{3}{4}\ A.\ L^{3/4}\ \ K^{3/4}\left[\frac{I}{L}\ dL + \frac{I}{K}\ dk\right]$$

$$= \frac{3}{4}\ \ q\left[\frac{I}{L}\ dL + \frac{I}{K}\ \ dK\right]$$

$$= \frac{3}{4}\left[\frac{q}{L}\ dL + \frac{q}{K}\ \ dK\right]$$

$$= \frac{3}{4}\ \left[AP_L\ \ dL + A\,P_k\ \ d\ k\right]$$

AP_L = Average product of labour, AP_K = Average Product of Capital

v) A production function is given by $q = 3L^{2/3}\,K^{1/3}$ where $60 - 2\,L - K = 0$, q = output, L = Labour, K = Capital. Find the least cost combination of labour and capital.

Sol. Given production function

$$q = 3\ L^{2/3}\ \ K^{1/3} \tag{1}$$

$$\text{Where } 60 - 2\ L - K = 0 \tag{2}$$

From (1) & (2) we have

$$Z = 3\ L^{2/3}\ K^{1/3} + \Upsilon\,(60 - 2\ L - K) \tag{3}$$

$$\frac{\partial Z}{\partial L}\ \ 3.\ \frac{2}{3} L^{\frac{-1}{3}}\ K^{\frac{1}{3}}\ - 2\lambda = 2 L^{\frac{-1}{3}}\ K^{\frac{1}{3}}\ - 2\lambda = 0 \tag{4}$$

$$\frac{\partial Z}{\partial K} = 3.\ L^{\frac{2}{3}}\ \frac{1}{3}\ K^{\frac{-2}{3}} - \lambda = L^{\frac{2}{3}}\ \ K^{\frac{-2}{3}}\ - \lambda = 0 \tag{5}$$

$$\frac{\partial Z}{\partial \lambda}\ =\ 60 - 2L - K = 0 \tag{6}$$

From (4) and (5), $2L^{-1/3}\ K^{1/3} = 2\lambda$, $L^{2/3}\ \ K^{-2/3} = \lambda$

Dividing, we get

$$\frac{L^{\frac{-1}{3}}\ \ K^{\frac{1}{3}}}{L^{\frac{2}{3}}\ \ K^{\frac{2}{3}}}\ =\ \frac{\lambda}{\lambda}$$

$$K\ =\ L$$

Substitute K = L, in (6), 60-2L - L = 0, 3L = 60 L = 20 $\therefore$ K = 20

When L = 20, K = 20, Then the first condition of tangency is satisfied

For Second condition $\dfrac{d^2\ K}{d\ L^2}\ > 0$

$$\frac{-f_L}{f_k}\ =\ -\ \frac{2\ L^{\frac{-1}{3}}\ \ K^{\frac{1}{3}}}{L^{\frac{2}{3}}\ \ K^{\frac{-2}{3}}}\ =\ -2L^{-1}\ \ K$$

$$\frac{d}{d\ L}\ (-2L^{-1}\ \ k) = 2\ L^{-2}\ k\ +\ (-2L^{-1})\ \frac{dk}{dL}$$

$$=\ 2\ L^{-2}\ \ K - 2\ l^{-1}\ (-2\ \ L^{-1}\ \ k)$$

$$=\ 6\ L^{-3}\ \ K\ >\ 0$$

$\therefore$ output is maximum, when L = 20, K = 20

Examples of total derivative

i) If U = f (x, y) and y = f(x) = 4x-7, find the total derivative $\frac{\partial u}{\partial x}$ of the function.

$U = x^3 + 3x^2 y + 6 xy^2 + 2 y^3$

Sol : If U = f (x, y) and y = f (x), then the total derivative $\frac{\partial u}{\partial x}$ is given by

$$\frac{dU}{dx} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dx}$$

$$y = f(x) = 4x - 7$$

$$\therefore \frac{dy}{dx} = 4.$$

$U = x^3 3x^2 y + 6 xy^2 + 2y^3$

$$\frac{\partial u}{\partial x} 3x^2 + 6xy + 6y^2$$

$$\frac{\partial u}{\partial y} = 3x^2 + 12xy + 6y^2$$

$$\therefore \frac{du}{dx} = (3x^2 + 6xy + 6y^2) + (3x^2 + 12xy + 6y^2)4$$

$$= 3x^2 + 6xy + 6y^2 + 12x^2 + 48\ xy + 2yy^2$$

$$= 15x^2 + 54\ xy + 30y^2$$

Example :- U = $(x^2 + y^2)\sqrt{x+y}$ and y = 2 - 3x. find the total derivative (du/dx)

Sol. y = 2 - 3x

$$\frac{dy}{dx} = -3$$

$$U = (x^2 + y^2)\sqrt{x+y}$$

$$\frac{\partial U}{\partial x} = (x^2 + y^2)\left[\frac{1}{2}(x + y)^{\frac{-1}{2}}\ 1\right] + (x + y)^{\frac{1}{2}}\ 2x$$

$$= \frac{x^2 + y^2}{2\sqrt{x+y}} + \sqrt{x+y}\quad 2x$$

$$= \frac{x^2 + y^2 + 4(x+y)x}{2\sqrt{x+y}}$$

$$= \frac{x^2 + y^2 + 4x^2 + 4xy}{2\sqrt{x+y}}$$

$$= \frac{5x^2 + 4xy + y^2}{2\sqrt{x+y}}$$

$$\frac{\partial U}{\partial y} = (x^2 + y^2)\ \frac{1}{2}\ (x+y)^{\frac{-1}{2}}\ 1 + (x^2 + y)^{\frac{1}{2}}\quad 2xy$$

$$= \frac{x^2 + y^2}{2\sqrt{x+y}} + 2y\quad \sqrt{x+y}$$

$$= \frac{x^2 + 4xy + 5y^2}{2\sqrt{x+y}}$$

$$\frac{dy}{dx} = \frac{5x^2 + 4xy + y^2}{2\sqrt{x+y}} + \frac{x^2 + 4\ xy + 5y^2}{\sqrt{x+y}}\quad (-3)$$

$$= \frac{1}{2\sqrt{x+y}}\left[5x^2 + 4xy + y^2 - 3x^2 - 12xy - 15y^2\right]$$

$$= \frac{1}{2\sqrt{x+y}}\left[2x^2 - 8xy - 14y^2\right]$$

$$= \frac{1}{\sqrt{x+y}}\left[x^2 - 4xy - 7y^2\right]$$

Example:- Find the total derivative $\left(\frac{dU}{dx}\right)$ if U = f (x, y ; t) where

x = a + b t y = c + d t

Sol. U = f (x, y ; t, x = a + b t , y = c + d t

$$\therefore \frac{dU}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{dx}{dt} = b, \quad \frac{dy}{dt} \quad \partial$$

$$Also \ \frac{\partial u}{\partial x} = f_x \ (x, y, t), \ \frac{\partial u}{\partial y} = f_y \ (x, y, t) : 1$$

$$\therefore \ \frac{\partial u}{\partial t} = f_x \ (x,y,t)\ b + f\ y\ (x, y, t)\ d$$

$$= f_x \ b + f_y \ d$$

ii) If U = x^2 - 8xy - y^3, x = 3t, y = 1 - t

We know $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$\frac{\partial x}{\partial t} = 3, \quad \frac{\partial y}{\partial t} = -1$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 - 8xy - y^3) = 2x - 8y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - 8xy - y^3) = -8x - 3y^2$$

Hence $\frac{du}{dt} = (2x-8y)\ (3) + \left[-(8x+3y^2)\ (-1)\right]$

$-3\ (2x - 8y) + 8x + 3y^2$

$= \ 6x \ - \ 24y \ + \ 8x \ + \ 3y^2$

$= \ 14x \ - \ 24y \ + \ 3y^2$

Self-Assessment - III

1. If z = $\frac{x-y}{x+1}$, find the total derivative.

2. If z = $\frac{9x-7y}{2x+5y}$ and y = 3x - 4, find the total derivative.

6.6 SUMMARY

We conclude this lesson by summarizing what we have covered in it

i) Partial derivative:- Finding the first order partial derivative of a function U = f(x, y) w. r. t. x we assume y remains fixed and the change in the function is due to change in x and denote it by $\frac{\partial u}{\partial x}$ or f(x), similarly if x remains fixed to the change in the function is due to change in y, then $\frac{\partial u}{\partial y}$ or f(y)

ii) Second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} \text{ or } U_{xx} \text{ or } f_{xx} \text{ or } f_{11}$$

$$\frac{\partial^2 f}{\partial x^2} \text{ or } U_{yy} \text{ or } f_{yy} \text{ or } f_{22}$$

iii) Second order cross partial derivatives

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial_y \partial_x} \text{ or } f_{xy} \text{ or } f_{12} \text{ or } U_{12}$$

$$\frac{\partial^2 f}{\partial_x \partial_y} = \frac{\partial^2 u}{\partial_x \partial_y} \text{ or } f_{yx} \text{ or } f_{21} \text{ or } U_{21}$$

iv) Total derivatives

$$\frac{\partial u}{\partial x} = f_x + f_y \frac{dy}{dx}$$

$$\frac{\partial u}{\partial y} = f_x \frac{dx}{dy} + fy$$

v) Total differentials

$$U = \frac{\partial U}{\partial x} dx + \frac{\partial u}{\partial y} dy = f_x \, dx + f_y \, d_y$$

vi) Economic examples / application of partial derivatives, total derivatives and total differential.

6.7 LESSON END EXERCISE

Q.1. Find the total differential of the following functions

i) $\frac{x^2 + y^2}{x - y}$ ii) $\frac{1}{\sqrt{x^2 + y^2}}$

Q.2. Find du, when (i) $u \quad e^{2x^2 + 3y^2}$

(ii) $= Log \ (3x^2 - y^2)$

Q.3. Find dq, if the production function be $q = a \, L^{3/4}, \ C^{3/4}$

Q.4. Find the total differential of

$U = x^3 + 3x^2y + 6xy^2 + 2y^3$

6.8. SUGGESTED READINGS

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MAXIMA AND MINIMA - CONSTRAINED AND UNCONSTRAINED, ECONOMIC APPLICATION

STRUCTURE

7.1 INTRODUCTION

When we first introduced the term equilibrium in the first unit, we made a broad distinction between goal and non goal equilibrium. In the latter type, exemplified by our study of market and national-income models, the interplay of certain opposing forces in the model-e.g., the forces of demand and supply in the market models and the forces of leakages and injections in the income models- dictates an equilibrium state, if any, in which these opposing forces are just balanced against each other, thus obviating any further tendency to change. The attainment of this type of equilibrium is the outcome of the impersonal balancing of these forces and does not require conscious effort on the part of anyone to accomplish a specified goal. True, the consuming households behind the forces of demand and the firms behind the forces of supply are each striving for an optimal position under the given circumstances, but as far as the market itself is concerned, no one is aiming at any particular equilibrium price or equilibrium quantity. Similarly, in national income determination, the impersonal balancing of leakages and injections is what brings about an equilibrium state, and no conscious effort at reaching any particular goal needs to be involved at all.

In the present lesson, however, our attention will be turned to the study of goal equilibrium, in which the equilibrium state is defined as the optimum position for a given economic unit (a household, a business firm, or even an entire economy) and in which the said economic unit will be definitely striving for attainment of that equilibrium. As a result, in this context-but only in this context-our earlier warning that equilibrium does not imply desirability will become irrelevant and immaterial.

7.2 OBJECTIVES

After reading this lesson you should be able to understand

- Increasing and decreasing functions
- Convexity of curves
- Maxima and minima of functions without constraints.
- Maxima and minima of functions with constraints.
- Economic application.

7.3 INCREASING AND DECREASING FUNCTIONS

Let us assume x is output and y is average cost. Furthermore, let us assume that the relation between x and y is $y = 40-6x+x^2$. This can be thought of as an average cost function. Now, the question is, does y (average cost) increase decrease or stay stationary as x (output) increase. Let us first draw a graph (figure 7.1) of this cost function, which shows that as

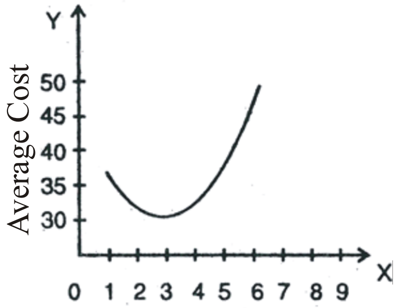

Fig. 7.1

x (output) increases, y (average cost) decreases, reaches a minimum, and then starts to increase. Graphing the first part of the function, we have figure (7.2). A function (curve) that is downward-sloping like this is called a decreasing function, that is, the value of the function y decreases as x increases. We shall show this mathematically, selecting a point A_1 on the curve and drawing a tangent to the curve. The slope of this tangent is given by tan α.

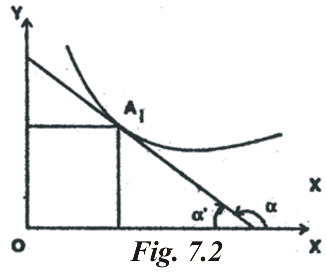

Fig. 7.2

We also know that the derivative of the function at the point A1, is equal to the slope of the curve at that point. Thus

$$\frac{dy}{dx} = \tan\alpha$$

But we see graphically that $\alpha > 90^0$. We know from trigonometry that

$$\tan \alpha = \tan (180^0 - \alpha') = \tan \alpha'$$

For example, if $\alpha = 135$, then $\alpha' = 45$. Thus,

$\tan 135^0 = \tan(180^0 - 45^0) = -\tan 45^0 = 1$ [$\because \tan 45^0 = 1$]

and the derivative will be equal to

$$\frac{dy}{dx} = \tan\alpha = \tan 135^0 = -1 < 0$$

In general, when $\alpha > 90^0$, then $\frac{dy}{dx} < 0$

As a conclusion, we may say that y = f(x) is a decreasing function at the point x when f (x) <0. By a similar argument we may say that y =f(x) is an increasing function at the point x when f (x) >0

Example : Using our average cost function, let us find whether it is a increasing or decreasing function at the values x = 4 and x = 2.

a) When x = 4, $y = 40 - 6x + x^2$

$$\frac{dy}{dx} = 6 + 2x$$

$$\left(\frac{dy}{dx}\right)_{x=4} = -6 + 8 = 2$$

b) When x = 2

$$\left(\frac{dy}{dx}\right)_{x=2} = -6 + 4 = -2$$

Thus, at x = 2, it is decreasing (cost) function. This is interpreted as follow. When x increase by a small amount, y will decrease by 2 units, at the point x = 2.

In case of (a), at x = 4, it is an increasing function. When f' (x) = 0, the slope of the curve is horizontal. That is, $\tan\alpha = \tan 0^0 = 0$. At this point the curve is stationary.

Example 2 :

$y = x^2 - 4x$

$$\therefore \frac{dy}{dx} = 2x - 4$$

When x = -1, then f' (x) = -6 < 0. Thus, y = f(x) is a decreasing function at the point where x = -1

When x = 2, then f' (x) = 0 and then y = f(x) is a stationary function at x=2.

When x=3, then f' (x)=6-4= 2>0. Thus, y =f(x) is an increasing function at this point.

7.3.1 Convexity of Curves

Consider a car that starts from a stand still position and reaches a certain speed after a certain amount of time. Let y (meters) be the distance the car has travelled and t (seconds) be time. We have the following relationship.

$$y = t^2 \qquad \text{........ (1)}$$

Thus, when t = 1 second, then y = 1 meter; t = 3 seconds, then y = 9 meters, and so forth.

Let us differentiate with respect to t. Then,

$$\frac{dy}{dx} = 2t \qquad \text{......(2)}$$

dy/dt gives the velocity at time t. For example, when t = 2, i.e. 2 seconds after the car has started, dy/dt = 4 meters / second, and so forth.

Let us graph equation (1) dy/dt = 2t shows the slope of this curve at various points. For example, at the point (3, 9) we have

$$\frac{dy}{dx} = \tan\alpha = 2x3 = 6$$

$$\alpha = 80^0\ 30' \text{ (approximately)}$$

Now, let us differentiate (2) once more

$$\frac{d^2y}{dt^2} = 2 \qquad ....(3)$$

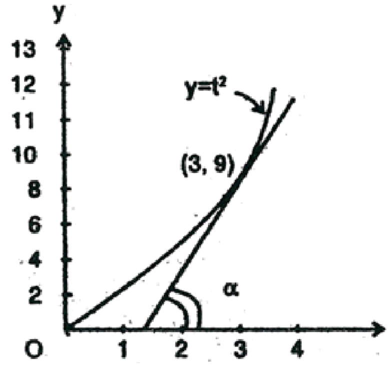

Fig. 7.3

This gives us the change in velocity per second, i.e., acceleration. We have 2 meters per second, as acceleration. This means velocity increases every second by 2 meters per second. Summarizing this in table form, we have

TABLE 1

t(seconds)	0	1	2	3	4
y(meters)	0	1	4	9	16
dy/dt (meters/second)	0	2	4	6	8
d^2y/dt^2 (meters/second)	2	2	2	2	2

We are interested in the second derivative $d^2y/dt^2= f''(t)$. This shows the rate of change. In our present case $f''(t) = 2$ml second^2 and it is constant, that is, the rate of change is the same. Thus, the velocity is increasing at a rate of 2m/ sec every second whether it is o ne second or four seconds after the start.

Using this discussion, we may define several terms. The function y = f(x) in figure 4(a) is our present case and we shall say y is increasing at an increasing rate. This is shown by

$$\frac{d^2y}{dx^2} = f''(x') > 0$$

The curve y=f(x) lies above the tangent, and we say the curve is concave upward or convex downward.

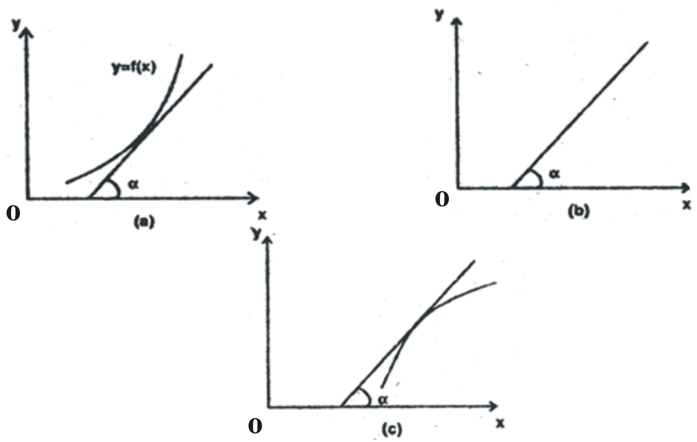

Fig. 7.4

The implication of $f''(x) > 0$ is that $f'(x) = \tan \infty$ is increasing at the point of tangent. For $f(x) = \tan \propto$ to increase, needs to increase towards 90^0. This can happen only if the curve gets steeper. Thus, the curve will be convex downwards as in figure 4(a)

Figure 4(b) shows where the rate of increase of y is zero. That is,

$$\frac{d^2y}{dx^2} = f''(x) = 0$$

For example, if

$$y = 2t, \frac{dy}{dt} = 2, \frac{d^2y}{dt^2} = 0$$

There is no curvature of the curve. The curve is a straight line.

Figure 4(c) shows where the rate of increase of y is decreasing. This is written

$$\frac{d^2y}{dt^2} = f''(x) < 0$$

To summarize, the condition for convex downward is f"(x)> 0, straight line, f" (x) = 0; and convex upwards, f"(x) < 0. In the special case where we have two curves tangent to each other, we can draw a tangent going through the point P where they touch. Then the curve APB will be above and the curve CPD will be below the tangent line.

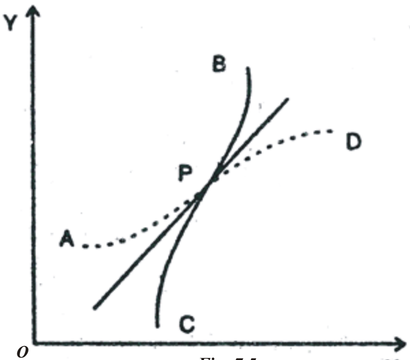

Fig. 7.5

If we look upon CPB as a curve and imagine the point P moving along the curve, then, when P is between C and D of the curve; f" <0 because CD is convex upward. But, from point P to B, since APB is convex downward, f" >0. Thus, at point P, f" changes signs and at that point, f"0. This point P is called the **point of inflexion**.

Example 1

$y = x^2-4x,\ f = 2x-4,\ f'' = 2>0$

Thus convex downward

Example 2

$$y = \frac{1}{2}x^4 - 3x^2, f = 2x^3 - 6x, f'' = 6x^2 - 6$$

The point of inflexion is where f"(x) = 0.

Thus, let

$6x^2-6 = 0, \qquad x^2-1 = 0,\ (x+1)\ (x-1) = 0$

$\therefore\ x = 1, \qquad x = -1$

Thus, the point of inflexion will be at

x = 1 and x = -1, For x = 1, we have

$$y = \frac{1}{2}x^4 - 3x^2$$

$$= \frac{1}{2} - 3 \qquad = -\frac{5}{2} = -2.5$$

and one of the points of inflexion is at (1, -2.5)

7.3.2 Maxima and Minima of functions of one Variable

i) Maxima and Minima of functions

Referring to figure 7.6, we see that points on either side of B_1 in a small neighbourhood are lower than B_1. Then, the function y = f(x) has a maximum value at the point B_1. In a similar value we define B_2 as the point where the function y = f(x) has a minimum value. The maximum and minimum values together are called the extreme values of the function. As can be seen, as the domain of x is enlarged, other maximum and minimum values may occur at different points. To emphasize that B_1 and B_2 may not be the only extreme values, we sometimes say relative maximum or minimum values or relative extreme values.

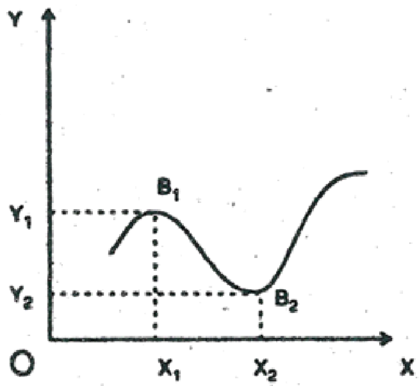

Fig. 7.6

Symbolically, this may be shown as follows:

Let $\in > 0$ be a small number, Then,

$f(x_1 - \in) < f(x_1) > f(x_1 + \in)$ …..Maximum

$f(x_2 - \in) < f(x_2) > f(x_2 + \in)$ .....Minimum

ii) Necessary and sufficient conditions for an extreme value

Assume we have two functions as shown in figures 7.7(a) and 7.7(b). From our previous discussion we know that when dy/dx = 0, the tangent to the curve at that point will be parallel to the x-axis. Let A and

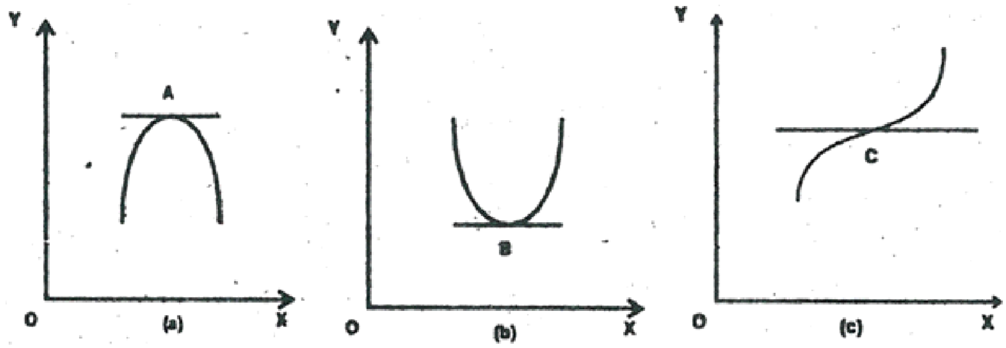

Fig. 7.7

B be two such point, then the tangents are parallel to the x-axis. As is evident from the graph, point A is a maximum and point B a minimum. In other words, the necessary conditions for a certain point such as A or B to be an extreme value is that dy/dx = 0 at that point.

From our discussion of the convexity of curves, however, we may have a situation such as in figure 7.7(c) where the tangent going through the point of inflexion is parallel to the x-axis. Thus, dy/dk=0 is a necessary condition for y =f(x) to have an extreme value but it is not a necessary and sufficient condition. Therefore, two questions arise: (1) how may we be certain that we do not have a point of inflexion such as 7.7(c) and (2) how do tell whether it is maximum or minimum? For this we can use our knowledge of the curvature of curves. Looking at figure 7.8 below we see that, for the left side of the curve, we have an increasing curve, that is, dy/dx>0. But we also see that it is increasing at a decreasing rate. That is, $d^2y/dx^2<0$, and thus the slope dy/dx must gradually approach zero.

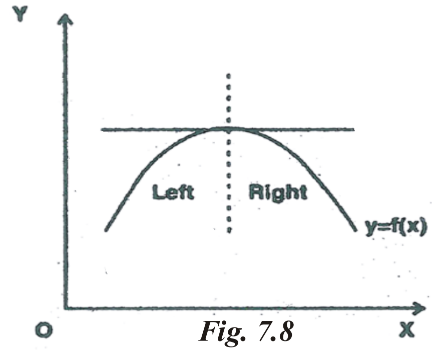

Fig. 7.8

Looking on the right side of the curve, we see it is a decreasing curve. Thus $dy/dx<0$. But we see it is decreasing at an increasing rate. Since $dy/dx<0$ this mean that $d^2y/dx^2<0$. Putting the two sides together, the curvature of this curve is shown by $d^2y/dx^2<0$. So, when we have a point that satisfies.

$$\frac{dy}{dx} = 0,\ \frac{d^2y}{dt^2} < 0$$

We must have a maximum, $dy/dx = 0$ and $d^2y/dx^2<0$ are the necessary and sufficient conditions for the occurrence of a maximum. Note that $d^2y/dx^2<0$ alone tells us the curvature of the curve. Only when we have the necessary condition $dy/dx<0$ does $d^2y/dx^2<0$ have meaning as a sufficient condition with respect to the occurrence of a maximum.

In a similar manner, we find that

$$\frac{dy}{dx} = 0,\ \frac{d^2y}{dt^2} > 0$$

are the necessary and sufficient conditions for a minimum.

As was just discussed, the necessary and sufficient conditions need to be considered together. But, in many cases in economics, the $d^2y/dx^2 \gtrless o$ needs to be discussed for its implications. We shall call this, and its corresponding part in functions of more than two variables, the sufficient conditions with the understanding of the preceding discussion.

Example: Our cost function was

$$y = 40-6x+x^2$$

Then the necessary condition for this to have an extreme value is

$$\frac{dy}{dx} = -6 + 2x$$

$$\frac{dy}{dx} = 0$$

$$\therefore -6 + 2x = 0$$

$$x = 3$$

Now, is the value of y a maximum or minimum at x = 3?

For this we determine the second derivative to find the curvature of the curve. This is

$$\frac{d^2y}{dt^2} = 2 > 0$$

Thus, the curve is convex downwards and the function has a minimum value at x = 3, that is, at x = 3, we have the minimum average cost which is

$$y = 40-6x + x^2 = 40-6x^3 + 3^2$$

$$= 40-18 + 9$$

$$= 31$$

7.3.3 Maximum and Minima - An Alternative Approach

An alternative way is to consider the change of signs of f(x)

Maximum : f'(x)=0, f'(x) changes from + to -

Minimum : f'(x)=0, f'(x) changes from - to +

A heuristic explanation of this is obtained by studying figure 7.6

Example 1. Find the maximum and minimum value of

$$2x^3-15x^2+36x+20$$

Sol : Let $y = 2x^3-15x^2+36x+20$

$$\frac{dy}{dx} = 6x^2 - 30x + 36$$

$$= 6(x^2 - 5x + 6)$$

For stationery value, $\frac{dy}{dx} = 0$

$$6(x^2 - 5x + 6) = 0$$

$$6(x-3)(x-2) = 0$$

Thus, stationary values occur at x=2 and x=3

Now, $\frac{d^2y}{dx^2} = 12x - 30$

For $\frac{d^2y}{dx^2} = 24 - 30 = -6 < 0$

At $x = 2, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$ Hence y is maximum and maximum value of y is

$$y = 2\,(2)^3 - 15\,(2)^2 + 36(2) + 20$$

$$= 2 \times 8 - 15 \times 4 + 72 + 20$$

$$= 16 - 60 + 72 + 20 = 48$$

At $x = 3, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$ Hence y is minimum and value is

$$y = 2\,(3)^3 - 15\,(3)^2 + 36(3) + 20$$

$= 54 - 135 + 108 + 20$

$= 47$

Example: Show that the maximum value of the function $y = x^3-27x+108$ is 108 more than the minimum value

Sol: Let $y = x^3-27x+108$

$$\frac{dy}{dx} = 3x^2 - 27$$

$$\therefore \frac{dy}{dx} = 0$$

$$3x^2 - 27 = 0$$

$$x^2 = 9$$

$$\therefore x = \pm 3$$

Now, $\frac{d^2y}{dx^2} = 6x$

For $x = \pm 3$

$$\frac{d^2y}{dx^2} = 18 > 0,$$

At $x = 3, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ Hence at $x = 3$, y is minimum and value is

$y = x^3 - 27x + 108$

$= 27 - 27(3) + 108$

$= 27 - 81 + 108$

$= 54$

At $x = -3, \frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ Hence at $x=-3$, y is maximum and max. value is

$$y = x^2 - 27x + 108$$

$$= (-3)^3 - 27(-3) + 108$$

$$= -27 + 81 + 108$$

$$= 162$$

$\therefore$ Max. value (162) is 108 more than min. value (54)

Self-Assessment - I

1. Find the maximum and minimum value of $(x - 2)^6 (x - 3)^5$

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2. Show that $x^3 - 3x^2 + 3x + 7$ has no maximum and no minimum value

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7.4 CONSTRAINED OPTIMIZATION WITH LAGRANGE MULTIPLIER

Differential calculus is also used to maximize or minimize a function subject to constraint given a function f (x, y) subject to a constraint I (x, y) = k (constant), a new function F can be formed (1) setting the constraint equal to zero (2) multiplying it by λ (the Lagrange multiplier, and (3) adding the product to the original equation

$$F(x, y, \lambda) = f(x, y) + \lambda\ [k - g(x, y)]$$

Here $F(x, y, \lambda)$ is the Lagrangian function, f (x, y) is the original or objective function, and g (x, y) is the constraint. Since the constraint is always set equal to zero, the product $\lambda[k-g(x,y)]$ also equals zero and the addition of the term does not change the value of the objective function. Critical values x_0, y_0 and λ_0, at which the function is optimized, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero, and solving simultaneously.

$$F_x(x, y, \lambda) = 0; \qquad F_y(x, y, \lambda) = 0; \qquad F_\lambda(x, y, \lambda) = 0$$

Second order conditions differ from those of unconstrained optimization and are treated as under. The second order conditions can now be expressed in terms of a bordered Hessian $|\bar{H}|$

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{yy} \end{vmatrix}$$

which is simply, the plain Hessain $\begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$ bordered by the first derivative of the constraint with zero on the principle diagonal. The order of a bordered principle minor is determined by the order of the principle minor being bordered.

Hence $|\bar{H}|$ above represents a second bordered principal minor $|\bar{H}_2|$, because the principal minor being bordered is 2x2.

For a function in n variables $f(x_1, x_2.., x_n)$ subject to $g(x_1, x_2.., x_n)$

$$|\bar{H}| = \begin{vmatrix} F_{11} & F_{12} & F_{1n} & g_1 \\ F_{21} & F_{22} & F_{2n} & g_2 \\ ---- & ---- & ---- & ---- \\ F_{n1} & F_{n2} & F_{nn} & g_n \\ g_1 & g_n & g_n & 0 \end{vmatrix} \text{ or } \begin{vmatrix} 0 & g_1 & g_2 & g_n \\ g_1 & F_{11} & F_{12} & F_{1n} \\ ---- & ---- & ---- & ---- \\ g_2 & F_{21} & F_{22} & F_{2n} \\ g_n & F_{n1} & F_{n2} & F_{nn} \end{vmatrix}$$

where $|\bar{H}| = |\bar{H}_n|$, because of the nxn principle minor being bordered.

If all the principle minors are negative i.e. if $|\bar{H}_2|, |\bar{H}_3|$ -- $|\bar{H}_4| < 0$, the bordered definite Hessian always satisfy the sufficient condition for a relative minimum.

If the principle minors alternate consistently in sign from positive to negative i.e., if $|\bar{H}_2| > 0$, $|\bar{H}_3| < 0$, $|\bar{H}_4| > 0$, etc. the bordered Hessian is negative definite and a negative definite Hessian always meets the sufficient condition for a relative maximum.

Example : Optimize the function, $2 = 4x^2 + 3xy + 6y^2$ subject to the constraint $x + y = 56$

Solution: Setting the constraint equal to zero

$$56 - x - y = 0 \qquad \text{---- (1)}$$

Multiply this difference by λ and add the product of the two to the objective functions in order to form the Lagrangian function Z

$$Z = 4x^2 + 3xy + 6y^2 + \lambda\,(56-x-y) \qquad \text{---- (2)}$$

$$Z_x = 8x + 3y - \lambda = 0 \qquad \text{---- (3)}$$

$$Z_y = 3x + 12y - \lambda = 0 \qquad \text{---- (4)}$$

$$Z_\lambda = 56 - x - y = 0 \qquad \text{---- (5)}$$

Subtracting (4) from (3), to eliminate λ, gives

$$5x - 9y = 0$$

$$5x = 9y$$

$$x = \quad , y = 1.8y$$

Substitute in (5) $\frac{9}{5}$

$$56 - x - y = 0$$

$$56 - 1.8y - y = 0$$

$$2.8y = 56$$

$$y = 20$$

we can also find x = 36

Substitute the critical values in (2)

$$Z = 4(36)^2 + 3(36)(20) + 6(20)^2 + 348(56-36-20)$$

$$= 4(1296) + 3(720) + 6(400) + 348(0)$$

$$= 9744$$

Also, we can find

$Z_{xx} = 8$, $Z_{yy} = 12$, $Z_{xy} = Z_{yx} = 3$

From the constraint, x + y = 56, we get $g_x = 1$ and $g_y = 1$. Thus,

$$|\bar{H}| = \begin{vmatrix} 8 & 3 & 1 \\ 3 & 12 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$|\bar{H}_2| = |\bar{H}| = 8(-1) + 3(1) + 1(3-12)$$

$$= -8 + 3 - 9$$

$$= -14$$

with $|\bar{H}_2| < 0 |\bar{H}|$ is positive definite, which means that Z is at a minimum

Example : The generalized Cobb-Douglas production function $q = K^{0.4} L^{0.5}$, given a budget constraint of Rs. 108 when $P_K = 3$, $P_L = 4$

Solution : $Q = (K^{0.4})(L^{0.5})$

$Q_K = 0.4(k)^{-0.6}, L^{0.5}$; $Q_L = 0.5, K^{0.4}, L^{-0.5}$

$Q_{KK} = (0.4)(-0.6)\ k^{1.6}, L^{0.5}$; $Q_{LL} = (0.5)(-0.5)\ K^{0.4}, L^{-1.5}$

$Q_{KK} = -0.24\ K^{1.6}, L^{0.5}$; $Q_{LL} = -0.25, K^{0.4}, L^{-1.5}$

Also $Q_{KL} = Q_{LK}$ $= (0.4)(0.5)\ K^{-0.6}, L^{-0.5}$

$= 0.20\ K^{-0.6}, L^{-0.5}$

and from the constraint 3 k + 4 L = 108

$$g_K = 3,\ g_L = 4$$

$$|\bar{H}| = \begin{vmatrix} -0.24K^{1.6}\ L^{0.5} & 0.20K^{-0.6}\ L^{-0.5} & 3 \\ 0.2K^{-0.6}\ L^{-0.5} & -25K^{0.4}\ L^{-1.5} & 4 \\ 3 & 4 & 0 \end{vmatrix}$$

Starting with $|\bar{H}_2|$ and expanding along the third row

$$|\bar{H}| = 3(0.8K^{-0.6}\ L^{-0.5} + 0.75K^{-0.4}\ L^{-1.5})$$
$$-4(-0.96K^{-1.6}\ L^{0.5} - 0.6K^{-0.6}\ L^{-0.5})$$

$$= 2.25K^{0.4}\ L^{-1.5} + 4.8K^{-0.6}\ L^{-0.5} + 3.84K^{-1.6}\ L^{0.5}$$

$$= \frac{2.25K^{0.4}}{L^{1.5}} + \frac{4.8}{K^{0.6}\ L^{0.5}} + \frac{3.84L^{0.5}}{K^{1.6}} > 0$$

with $|\bar{H}_2| > 0$; $|\bar{H}|$ is negative definite and Q is maximized

Examples of constrained maxima and minima

Example : Find the extremum (if any) of the function by Lagranges method

$U = f(x, y) = 12xy - x^2 - 3y^2$ subject to the constraint $x + y = 16$

Solution : The given objective function is

$$U = f(x, y) = 12xy - x^2 - 3y^2 \qquad \text{--- (1)}$$

and the constraint is $\phi(x, y) = x + y - 16$ --- (2)

$\therefore$ The Lagrange's function is

$$F = F(x, y; \lambda) = f(x, y) - \lambda\, \phi(x, y) \qquad \text{--- (3)}$$

$$= 12xy - x^2 - 3y^2 - \lambda(x+y-16) \qquad \text{--- (4)}$$

a) First order conditions

$$\left.\begin{aligned} F_x &= 12y - 2x - \lambda = 0 \\ F_y &= 12x - 6y - \lambda = 0 \\ F_\lambda &= -(x + y - 16) = 0 \end{aligned}\right\} \text{ or } \begin{aligned} 12y - 2x &= \lambda \quad ---(5) \\ 12x - 6y &= \lambda \quad ---(6) \\ x + y \quad &= 16 \quad ---(7) \end{aligned}$$

from (5) and (6)

$12y - 2x = 12x - 6y$

or $14x = 18y$

or $7x = 9y$ or $y = 7/9x$ --- (8)

Putting this value of y in (7), we get

$$x + \frac{7}{9}x = 16$$

$$\frac{16}{9}x = 16 \quad \text{or} \quad x = 9$$

from (8), y = (7)

From (5), $12(7) - 2(9) = \lambda$; $\lambda = 66$

Hence, there is one critical point $(x^*, y^*) = (9,7)$ and $\lambda = 66$

b) Second order conditions (SOC) $d^2F \gtrless 0$ for max/min

since $\phi(x, y) = x + y - 16$, $\phi x = 1$, $\phi y = 1$

$F_{xx} = -2$, $F_{yy} = -6$, $f_{xy} = 12 = f_{yx}$

Applying bordered Hessian Determinant (BHD) method

At the C.P. (9, 7), $\phi_1 = 1$, $\phi_2 = 1$, $F_{11} = -2$, $F_{12} = 12 = F_{21}$ $F_{22} = -6$

$$\therefore \ \Delta_3\left[\text{or}\left|\overline{H}_2\right|\right] = \begin{bmatrix} 0 & \phi_1 & \phi_2 \\ \phi_1 & F_{11} & F_{12} \\ \phi_2 & F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 12 \\ 1 & 12 & -6 \end{bmatrix}$$

$= 0 (\ \ ) -1 [(-6-12) + 1(12+2)]$

$= -1 (-18) + 1(14)$

$= 18 + 14$

$= 32 > 0$

which $\Rightarrow$ that the function $U = f(x, y) = 12xy - x^2 - 3y^2$ subject to the constraint

$\phi (x,y) = x + y - 16 = 0$ is maximum at the C.P (9, 7) and the constrained maximum value = Max $f = f(9, 7) = 12(9)(7) - (9)^2 - 3(7)^2$

$= 756 - 81 - 147$

$= 756 - 228 = 528$

Self-Assessment - II

1. Find whether the function is relative minima or maxima. If

   a) $z = 3x^2 - xy + 2y^2 - 4x - 7y + 12$

   b) $z = 2y^3 - x^3 + 147x - 54y + 12$

7.5 APPLICATION OF MINIMA AND MAXIMA IN ECONOMIC THEORY

Maxima and Minima is a potent device with the students of economics confronting myriads of problems in economic theory. It is a common experience that a rational consumer thinks always in terms of maximum utility, whereas a prudent producer strives at maximizing profits and choosing the least cost combination. As explained in the previous sub-unit, the technique of determining maximum and minimum value of a function of one variable. Even this limited knowledge will suffice to illustrate the use of this technique in solving many economic problems.

1.a) Maximum Revenue

When the demand function takes the form $P = K - aQ$, then

$TR = KQ - aQ^2$

This is a relation between TR and Q and TR will be at a maximum when the first and second-order conditions hold, i.e. when

$$d(TR)/dQ = 0$$

and $$d^2(TR)/dQ^2<0$$

when $$\frac{d(TR)}{dQ} = 0,$$

$$K\text{-}2a\ Q = 0 \text{ or } Q = K/2a$$

$$d^2(TR)/dQ^2 = -2a < 0$$

Thus TR is maximum when Q = K/2a

1.(b) Maximum Revenue and Elasticity

The first order condition for a maximum is d (TR)/dQ =0 or MR=0, TR remains unchanged as Q changes and E_D=-1

In the above example TR is at a maximum when Q = K/2a

$$E_D \frac{P}{Q}\frac{dQ}{dp}$$

$$= \frac{K - aQ}{Q}\left(-\frac{1}{a}\right)$$

Since P = K-aQ and dQ / dP = -1/a

When Q = K/2a

$$E_D = \frac{K - a\frac{K}{2a}}{K/2a}\left(-\frac{1}{a}\right)$$

$$= \frac{K - K/2}{K/2a}\left(-\frac{1}{a}\right)$$

$$= \frac{K/2}{K/2a}\left(-\frac{1}{a}\right)$$

$= a(-1/a) = -1$

2. Minimum Average Costs

If a firm's TC function is

$TC = K + aQ + bQ^2$, where K, a and b are positive constants then

$AC = TC/Q$

$$= \frac{K}{Q} + a + bQ$$

Average costs are at a minimum when $d(AC)/dQ = 0$ and $d^2(AC)/dQ^2 > 0$

$$d\frac{(AC)}{dQ} = -\frac{K}{Q^2} + b = 0$$

i.e. $Q^2 = \frac{K}{b}$

$Q = \sqrt{K/b}$

Second order condition

$$d^2\frac{(AC)}{dQ} = \frac{2K}{Q^3} > 0 \text{ for all } Q > 0$$

Thus AC is at a minimum when Q=

3. Minimum Marginal Costs $\sqrt{K/b}$

If a firm's TC function is

$TC = K + aQ - bQ^2 + cQ^3$, where K, a, b, and c are positive constant, then

$d(TC)/dQ = a - 2bQ + 3cQ^2 = MC$

MC is at a minimum when

$d(MC)/dQ = 0$

and $d^2(MC)/dQ^2 > 0$

when $d(MC)/dQ = 0$

$-2b+6c\,Q = 0$

$$Q = \frac{2b}{6c} = \frac{b}{3c}$$

$d(MC)^2/dQ^2 = 6c > 0$ c is positive

Thus MC is at a minimum when $Q = b/3c$

4.a) Profit Maximization

The conditions for a maximum derived can be used

Profit S = TR - TC

If $TC = K+aQ+bQ^2$ and the demand function takes the form $P = L-nQ$, where k, a, b, L and n are positive constants

$$TC = PxQ$$

$$= LQ - nQ^2$$

Then

$$S = LQ-nQ^2-K-aQ-bQ^2$$

$$= -(b+n)\,Q^2+(L-a)\,Q-K$$

S is now a function of Q and takes a maximum when

$dS/dQ = 0$ and $d^2S/dQ^2<0$ when $\frac{dS}{dQ} = 0$

$-2(b+n)\,Q+L-a=0$

$$Q = \frac{L-a}{2(b+n)}$$

$d^2S/dQ^2 = -2(b+n)<0$ since b and n are positive

Therefore profits are maximized when

$$Q=(L-a)/2\ (b+n)$$

4.b) The Profit Maximization Conditions

$$S = TR-TC$$

and $dS/dQ \ = \dfrac{d(TR)}{dQ} - \dfrac{d(TC)}{dQ}$

Since S, TR and TC are all functions of Q.

When $dS/dQ = 0$,

$$\frac{d(TR)}{dQ} - \frac{d(TC)}{dQ} = 0$$

or $MR-MC = 0$

Consequently, the first-order condition for a maximum is MR=MC

We know that revenue is maximized when MR= 0, or $E_D=-1$. A profit maximizing firm will be in equilibrium when MR = MC and since MC must be positive MR will take a positive value in equilibrium. Consequently, a profit maximizing firm will be in equilibrium when $-E_D>1$. When

$$= \frac{d^2(TR)}{dQ^2} - \frac{d^2(TC)}{dQ^2} \text{ or } \frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} \text{must be} < 0$$

Hence profits are maximized when MR=MC and d(MR)/dQ-d(MC)/dQ is negative. Figure (7.9) shows that profits are maximized at M. At M, MC = MR,

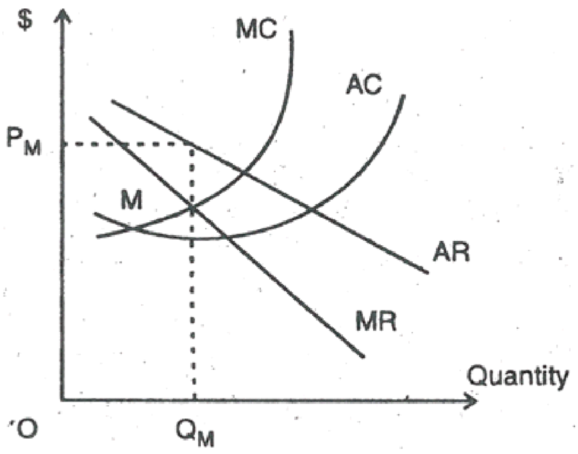

Fig. 7.9

MR is downwards sloping, i.e. the slope of the tangent to the MR curve is negative at every point, thus

d(MR)/dQ<0 at M

MC is upward sloping at M, thus

d(MR)/dQ>0

and $$\frac{d(MR)}{dQ} - \frac{d(MC)}{dQ} < 0$$

In general, however, we require only that d(MR)/dQ-d(MC)/dQ<0, so that d(MC)/dQ need not been positive.

5. Effect of Taxation on the output of a profit maximizing firm

Earlier in this unit, we saw that profits are maximized when

$$Q = \frac{L-a}{2(b+n)}$$

given that TC = K+aQ+bQ^2 and P=L-nQ. The levying of a lumpsum tax, i.e. a tax which does not depend on output, on this firm will have no effect on the equilibrium output. This tax will increase the value of the constant K in the TC function and consequently decrease profits by the amount of tax. However, the equilibrium output will remain unchanged since the constant term disappears with differentiation.

A tax which varies with output will affect the equilibrium output and profits. If the government imposes a tax of t per unit of quantity produced then TC will increase by tQ or S will decrease by tQ, i.e.

$$S = -(b+n)\ Q^2 + (L-a)\ Q-K-tQ$$

$$= -(b+n)\ Q^2 + (L-a-t)\ Q-K$$

when $dS/dQ = 0$

$$\frac{dS}{dQ} = -2(b+n)Q + (L-a-t) = 0$$

Thus $Q = \dfrac{L-a-t}{2(b+n)}$

$$\frac{d^2S}{dQ^2} = -2(b+n) < 0, \text{since } b+n \text{ are} + \text{ve}$$

Thus profits are maximized when

$Q = \dfrac{L-a-t}{2(b+n)}$. The optimum output falls as a result of the per unit tax: t is subtracted from the numerator and has the effect of decreasing Q. A per unit subsidy would have the opposite effect.

6. Maximization of Tax Revenue

A per unit tax on quantity produced causes a profit maximizing firm to cut back production. Since total tax revenue T depends upon the tax rate t and the output level Q, i.e. $T = tQ$, it is possible to find the tax rate which maximizes total tax revenue from the point of view of the exchequer.

In the previous example, equilibrium quantity depended on the tax rate t,

i.e., $Q = (L-a-t)/2\ (b+n)$

$$T = tQ = t.\ \left(\frac{L-a-t}{2(b+n)}\right)$$

$$= \frac{L-at-t^2}{2(b+n)}$$

We now have T as a function of t, alternatively T could be expressed as a function of Q.

Tax revenue T is maximized when dT/dt = 0 and $d^2T/dt^2<0$. When $\frac{dT}{dt} = 0$

$$\frac{dT}{dt} = \frac{t-a-2t}{2(b+n)} = 0$$

i.e. L-a-2t = 0

L-a=2t

$$t = \frac{L-a}{2}$$

$$\frac{d^{2T}}{dt^2} = -\frac{2}{2(b-n)} = -\frac{1}{b+n} < 0$$

Thus T is maximized when t=(L-a)/2

When t = (L-a)/2

$$Q = \frac{L-a-(L-a)/2}{2(b+n)}$$

$$= \frac{2L-2a-L-a}{4(b+n)}$$

$$= \frac{L-a}{4(b+n)}$$

T= tQ, thus

$$T = \left(\frac{L-a}{2}\right)\left(\frac{L-a}{4(b+n)}\right)$$

$$= \frac{(L-a)^2}{8(b+n)}$$

7. The supply function and a per unit tax

Consider an industry with the following demand and supply functions

$$P = K - \alpha Q_D$$

and $\quad P = -K_1 + \beta Q_S$

where K, K_1, $\propto$ and β are positive constants if the government imposes a tax of t per unit of quantity produced and P is the market price to consumers, then the effective price for producers is (P-t). Therefore, the supply function which includes the tax is $P - t = -K_1 + \beta Q_s$

or $P = -K_1 + \beta Q_s + t$

At equilibrium, $Q_S = Q_D$, so

$$K - \alpha Q = K_1 + \beta Q_s + t$$

$$Q = \frac{K + K_1 - t}{\alpha + \beta}$$

Equilibrium Q is reduced as a result of the tax. To find the tax rate which maximizes total tax revenue T one can express T as a function of t. Alternatively T can be expressed as a function of Q. At equilibrium

$$t = K + K_1 - (\alpha + \beta)Q$$

$$T = t\,Q$$

Thus

$$T/Q = K + K_1 - (\alpha + \beta)Q$$

$$T = KQ + K_1 Q - (\alpha + \beta)Q^2$$

T is at a maximum when $DT/dQ = 0$ and $d^2T/dQ^2 < 0$.

When $dT/dQ = 0$

$$\frac{dT}{dQ} = K + K_1 - 2(\alpha + \beta)Q$$

$$K + K_1 - 2(\alpha + \beta)Q = 0$$

i.e. $Q = \dfrac{K + K_1}{2(\alpha + \beta)}$

$$\frac{d^2T}{dQ^2} = -2(\alpha + \beta) < 0 \qquad \alpha + \beta \text{ are} + \text{ve}$$

Thus, T is maximized when

$$Q = (K + K_1)/2\,(\alpha + \beta)$$

Once Q is known, t and T can be found by substitution

The conditions for a maximum or minimum discussed above apply only to continuous and smooth, i.e. without sharp bends. Most of the functions dealth with in this unit are quadratics so that the relative maximum or minimum of any of these functions is also the absolute maximum or minimum. With functions of degree 3 the relative maximum or minimum will not be the absolute maximum or minimum.

CUBIC FUNCTIONS

$$Y = ax + bx^2 + cx^3$$

where $c>0$; thus

$$dy/dx = a + 2bx + 3cx^2$$

This function will have a maximum and minimum when

$$a + 2bx + 3cx^2 = 0$$

Assuming real roots, the values of x which satisfy this equation will only provide a relative maximum or minimum. There is no limit to the value of y in the upward or downward direction, i.e. no absolute maximum or minimum, since larger positive values for x will keep increasing y and larger negative values will keep decreasing y.

7.3.4 Numerical Exercises

1. Example: If $P = \frac{121}{q+4} - 1$. Find the output level at which total revenue is maximum also find maximum revenue.

Sol. $P = \frac{121}{q+4} - 1$

$$R = \frac{121xq}{q+4} - q$$

$$\frac{dR}{dQ} = \frac{484}{(q+4)^2} - 1$$

$$\frac{d^2R}{dq^2} = \frac{-968}{(q+4)^3}$$

For stationary values, $\frac{dR}{dq} = 0$

$$\therefore \frac{484}{(q+4)^2} - 1 = 0$$

$$484 = (q+4)^2$$

$$22 = q+4$$

$$q = 18$$

when q = 18, $\frac{d^2R}{dq^2}$ is $-$ve

$\therefore$ R is maximum at q = 18 and maximum revenue is

$$R = \frac{121x18}{22} - 18$$

$$= \frac{18[121-18]}{22}$$

$$= \frac{18x99}{22} = 81$$

2. Example: For a firm under perfect competition total cost function is given by $\pi = \frac{1}{25}q^3 - \frac{9}{10}q^2 + 10q + 12$. If the price is Rs. 4 per unit, will the firm continue production?

Sol. $MC = \frac{d\pi}{dq} = \frac{3}{25}q^2 - \frac{9}{5}q + 10$

The first order condition for a firm under perfect competition for profit maximization is MC = Price

i.e. $\frac{3q^2}{25} - \frac{9}{5}q + 10 = 4$

$$\frac{3q^2}{25} - \frac{9}{5}q + 6 = 0$$

$$\frac{q^2}{25} - \frac{3}{5}q + 2 = 0$$

$$q^2 - 15q + 50 = 0$$

$$\therefore \quad q = \frac{15 \pm \sqrt{(15)^2 - 4(1)(50)}}{2}$$

$$= \frac{15 \pm \sqrt{225-200}}{2} = \frac{15 \pm \sqrt{25}}{2}$$

$$= \frac{15 \pm 5}{2} = 5,10$$

The second order condition for profit maximization is

$$\frac{d^2\pi}{dq^2}\text{must be} + \text{ve}$$

Now

For $q = 5, d^2\pi/dq^2 \frac{6}{25}x5 - \frac{9}{5} = -\frac{3}{5}$ which is $-$ ve

$q = 10, d^2\pi/dq^2 \frac{6}{25}x5 - \frac{9}{5} = -\frac{3}{5}$ which is $+$ ve

Profits are maximum when q = 10

λ

At q = 10, Profit $= Pq - \left[\frac{1}{25}q^3 - \frac{9}{10}q^2 + 10q - 12\right]$

$$= 40 - \left[\frac{1}{25}(10)^3 - \frac{9}{10}(10)^2 + 10(10) + 12\right]$$

$$= 40 - [40 - 90 + 100 + 12]$$

$$= 40-[62] = -22$$

Maximum profit is a loss of Rs. 22. The fixed cost in this case is Rs. 12 and loss of more than the fixed cost. If the firm discontinues production when p=4, its loss is reduced to Rs. 12. Hence, the firm will produce nothing if p=4

Self-Assessment - III

1. Determine the minimum value of the cost function

   C = q2 - 4q + 100

2. Given p = 200 - .02q and c = 100q + 30000. A tax of Rs. 50 per unit of output is levied. Determine the quantity at which profit is maximum. Find the maximum profit and price.

7.6 SUMMARY AND CONCLUSION:

We conclude this lesson, by summarizing what we covered in it increasing and decreasing function, maxima and minima without constraints and maxima and minimum with constraints and application of minima and maxima in economic theory.

7.7 LESSON END EXERCISE

Q.1 State the conditions for the existence of maxima and minima of a function $y = f(x)$

Q.2 What do you mean by constraint maximization and also write the necessary and sufficient conditions of constrained optimization.

Q.3 Show that the demand curve

$y = 80-5x-x^2$ is downward sloping and concave from below.

Q.4 A fixed plant is used to manufacture radio sets and, if x sets are turned out per week, the total variable cost is

$$\text{Rs}\left(3x + \frac{1}{25}x^2\right)$$

Show that the average a variable cost increases steadily with output

Q.5 Show that the maximum value of $\left(\frac{1}{x}\right)^x$ occurs at $x = 1/2$

Q.6 Divide 24 into 2 parts, each part being positive real number, such that the product is maximum.

Q.7 Find the maximum or minimum values of $x^2 .e^{1/x2}$

Q.8 Find the minimum value of the function $U = x^2+3y^2+5z^2$ under the condition $2x+3y + 5z = 100$

Q.9 If $U = x^2y$ is a utility function find the equilibrium if budget constraint of $x + 2y = 4$

Q.10 Given the utility function $U = (x+2)(y+1)$ and the budget constraint $2x+5y = 51$, find the optimal levels of x and y purchases by the consumer.

7.8 SUGGESTED READINGS

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M.A. Economics | Lesson No. 8
C.No. 103 | Semester - 1st | Unit II

PRINCIPLE OF INTEGRATION INDEFINITE, DEFINITE, ECONOMIC APPLICATION

STRUCTURE

8.1 INTRODUCTION

The first limiting process discussed was differentiation. From a geometrical standpoint, differentiation was a study of the tangent of a curve. The second limiting process we take up is integration. In geometrical terms will be a study of area under a curve. Analytically speaking, however, integration and differentiation do not depend on geometry. A geometrical interpretation is used only to help foster intuition.

8.2 OBJECTIVES

After reading this lesson, you would be able to understand.

- Principles of integration
- Definite Integrals
- Indefinite integrals
- Economic application

8.3 DEFINITION

$$\mathrm{IF} = \frac{d}{dx} + [f(x)] = F(x)$$

Then f(x) is said to be integral of F(x) and is written as

$$\{F(x)\,dx\ f(x)\}$$

[Real a integral of F f(x) dx us f(x)]

The sign $\int$ is the integral sign while the function f(x) is the integrand (i.e. the function whose integral is to be found. The dx is added to indicate the variable with respect to which the function f(x) is to be integrated. It also acts as an indication of where the function to be integrated finishes.

Integration is thus the reverse of differentiation. Unlike differentiation, however, there is no general rule for integration. Indeed a function need not have a integral. However, most of the functions one finds in economics do have integrals. Knowledge of the results of differentiation is essential if one is to derive the integral of a function.

8.3.1 Standard Form of Integrals

Suppose we wish to find the integral of X^2; i.e. X^2 is the derivative of a function and we want to discover the function.

$$X^2 = f(X)$$
$$f(X) = \int f(X)\,dx = ?$$
We know that $dx^2/dx = 3x^2$, thus
$$d\left(x^3/3\right)\Big/dx = 3x^2/3 = x^2$$

Consequently the function whose derivative is x^2 equals $x^3/3$, $x^3/3$, is not integral of x^2 but an integral. Consider the function.

$$\frac{X^3}{3} + K$$

Where K is constant. The derivative of $(x^3/3)+K$ with respect to X is also X^2 . Hence the function whose derivative equals x^2 is $(x^3/3)+K$, where K is called the constant of integration. Further information is required before a definite value can be assigned to K.

The function whose derivative is X^{10} is given by

$$f(x) = \int X^{10} dx = \frac{X^{11}}{11} + K$$

Where K is the constant of integration

It is possible to check by taking the derivative of $(X^{11}/11+K$, with respect to x, e.g. $\frac{d}{dx}\left[\frac{X^{11}}{11} + K\right] = \frac{11X^{10}}{11} = X^{10}$

Suppose we wish to find the integral of X^n, i.e.

$$X^n = f(x)$$

then

$$f(x) = \int X^n dx = \frac{x^{n+1}}{n+1} + K$$

Because

$$\frac{d}{dx}\left\{\left[\frac{x^{n+F}}{n+1}\right]+K\right\}=n+1\frac{X^{n}}{n+1}=X^{n}$$

When n= 4, then

$$f(x)=\int X^{4}dx=\frac{X^{4+1}}{4+1}=K=\frac{X^{5}}{5}+K$$

When n = -2, then

$$f(x)=\int X^{-2}dx=\frac{X^{-2+1}}{-2+1}+K=-\frac{1}{x}+K$$

This works for all values of n except n = -1

1. INTEGRAL OF A CONSTANT

If the constant b is the derivative of a function, i.e. b = f(x), then.

$$f(x)=\int b.dx=bx+K$$

$$\text{because}\ \frac{d}{dx}[bx+K]=K$$

2. INTEGRAL OF SUM :

The integral of $X^{\alpha}+X^{\beta}+X^{\gamma}+b,$ where, α, β, γ and be are constant is

$$\int x^{\alpha}dx+\int x^{\beta}dx+\int x^{v}dx+\int b\ dx$$

$$=\frac{x^{\alpha+1}}{\alpha+1}+\frac{x^{\beta+1}}{\beta+1}+\frac{x^{r+1}}{\gamma+1}+bx+K$$

3. INTEGRAL OF MULTIPLE

The integral of a constant multiple of any integrand and is the constant times the integral of that integrand and, e.g.

$$\int b(x^{\alpha}+a)dx=b\int(x^{\alpha}+a)dx$$

$$=\left[\frac{x^{\alpha+1}}{\alpha+1}+ax+K\right]$$

Where k is the constant of integration

8.3.2 Some Standard Forms :

1. $\int \frac{1}{x} dx = \log x + k \left[\frac{x}{dx} (\log x + k) = 1/x \right]$

2. $\int a^x dx = \frac{ax}{\log a} + k \left[\because \frac{d}{dx} (a^x + k) = a^x \log a \right]$

3. $\int e^x dx = e^x k \left[\because \frac{d}{dx} (e^x + k) = e^x \right]$

1. Example : Integrate the following w.r.t. x

$$\frac{a+x}{x}$$

$$= \int \left(\frac{a+x}{x} \right) dx$$

$$= \int \left(\frac{a}{x} + 1 \right) dx = a \int \frac{1}{x} dx + \int 1 dx = a \log x + x$$

2. $\int \frac{1}{\sqrt{x-1} - \sqrt{x+1}} dx$

$$= \int \left[\frac{1}{\sqrt{x-1}\sqrt{x+1}} x \frac{\sqrt{x+1} + \sqrt{x+1}}{\sqrt{x-1} + \sqrt{x+1}} \right] dx$$

$$= \int \frac{\sqrt{x-1} + \sqrt{x+1}}{x-1-x-1} dx$$

$$= \frac{1}{2} \int \left(\sqrt{x-1} + \sqrt{x+1} \right) dx$$

$$= \frac{1}{2} [\int (x-1)^{1/2} dx + \int (x+1)^{12/} dx$$

$$= \quad -\frac{1}{2}\left[\frac{(x-1)^{3/2}}{3/2} + \frac{(x+1)^{3/2}}{3/2}\right]$$

$$= \quad -\frac{1}{2}\left[(x-1)^{3/2} + (x+1)^{3/2}\right]$$

A. INTEGRATION BY PARTIAL FRACTION

$$\int \frac{1}{x^2 - 16} dx$$

Let I = $\int \frac{1}{x^2 + 16} dx = \int \frac{1}{(x+4)(x-4} dx$

Now let

$$\therefore \frac{1}{x^2 - 16} = \frac{A}{x+4} + \frac{B}{x-4}$$

$$1 = A(x-4) + B(x+4)$$

$$\text{Put } x = 4, \text{we get } B = 1/8$$

$$x = -4, \text{we get } A = -1/8$$

$$\therefore \int \frac{1}{x^2 - 16} dx = \left[\frac{1}{8(x+4)} + \frac{1}{8(x-4)}\right] dx$$

$$= -\frac{1}{8}\int \frac{1}{(x+4)} dx + \frac{1}{8}\int \frac{1}{x-4} dx$$

$$= -\frac{1}{8}\log(x-4) + \frac{1}{8}\log(x-4)$$

$$= \frac{1}{8}\left[-\log(x+4) + \log(x-4)\right]$$

$$= \frac{1}{8}\left[\log(x+4)^{-1} + \log(x-4)^{1}\right]$$

$$= \frac{1}{8}\log\frac{x-4}{x+4}$$

B. INTEGRATION BY SUBSTITUTION

Suppose, we have to evaluate, $\int (x)dx$ if we put $x = \phi(z)$ and $dx = \phi'(z)dz$ and evaluate the resulting integral $\int\left(\frac{a+x}{x}\right)dx$ the process of integration becomes simple. Illustrating by taking a example.

Show that $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} (n \neq -1)$

Let ax+b = z differentiating both sides

a dx = dz

dx = dxfa

Now, $\int (ax+b)^n dx = \int {}^n z \frac{dz}{a} = \frac{1}{a}\int^n z\,dz$

$= \frac{1}{a}\frac{z^{n+1}}{n+1}$ provided $n \neq 1$

$= \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}$ $\quad\because Z = ax+b$

Evaluate $\int x\sqrt{2x+1}\,dx$

Let $\sqrt{2x+1} = z$

Squaring both sides

$$2x+1 = z^2$$

$$d2dx = 2zdz$$

$$dx = z.dz$$

From above, $x = \frac{z^2-1}{2}$

$$\int x\sqrt{2x+1}\,dx = \int \frac{z^2-1}{2} z.z.dz$$

$$= \frac{1}{2}\left[\int \left(z^2-1\right)z^2 dz\right]$$

$$= \frac{1}{2}\left[\frac{z^5}{5} - \frac{z^3}{3}\right]$$

$$= \frac{1}{2}\left[\frac{(2x+1)^{5/2}}{5} - \frac{(2x+1)^{3/2}}{3}\right]$$

$$= \frac{(2x+1)^{3/2}}{2}\left[\frac{2x+1}{5} - \frac{1}{3}\right]$$

$$= \frac{(2x+1)^{3/2}}{2}\left[\frac{6x+3-5}{15}\right]$$

$$= \frac{(2x+1)^{3/2}}{2}\frac{(6x-2)}{15}$$

$$= \frac{1}{15}(2x+1)^{3/2}(3x-1)$$

C. THREE IMPORTANT FORMS:

1. $\int [f(x)]^n f'(x)dx = \dfrac{[f(x)]^{n+1}}{n+1} \qquad (n \neq -1)$

2. $\int \dfrac{f'(x)}{f(x)} dx = \log f(x)$

3. $\int e^{f(x)}.f'(x)dx = e^{f(x)}$

EXAMPLES : Integrate w.r.t.x

1. $\int \frac{x}{\sqrt[3]{x^2+1}}$

$$x\left(x^2+1\right)^{-1/3} dx = \frac{1}{2}\int \left(x^2+1\right)^{-1/3} 2d.dx$$

$$\int \left\{\text{Form} \int [f(x)]^n . f^{'}(x).dx\right\}$$

$$= \frac{1}{2}\frac{\left(x^2+1\right)^{-\frac{1}{3}+1}}{-1/3+1}$$

$$= \frac{1}{2}\frac{\left(x^2+1\right)^{2/3}}{2/3} = \frac{3}{4}.\left(x^2+1\right)^{2/3}$$

2. $\int \frac{ax^{n-1}}{bx^n}.dx = a\int \frac{x^{n-1}}{bx^n+c}.dx$

$$= \frac{a}{bn}\int \frac{bn.x^{n-1}}{bx^n+c}.dx$$

$$\left[\text{Form} \int \frac{f^{'}(x)}{f(x)}.dx\right]$$

$$= \frac{a}{bn}\log\left(bx^n+c\right)$$

3. $\int x^2.e^{2x^3+4}dx$

$$= \frac{1}{6}\int 6x^2.e^{2x^3+4}.dx$$

$$\left\{Form \int f^{'}(x).e^{f(x)}dx\right\}$$

$$= \frac{1}{6}.e^{2x^3+4}$$

D. INTEGRATION BY PARTS

Integral of the product of two functions

= first function x integral of the second - Integral of (the derivative of first x Integral of second)

Example : $\int x^2.e^x.dx$

$= \quad x^2\int e^x dx - \int 2x.e^x dx$

Taking x^2 as the first function

$= \quad x^2.e^x - 2\int x.e^x dx$

Taking x as the first function

$= \quad x^2.e^x - 2\{x.\int e^x dx - \int e^x dx\}$

$= \quad x^2.e^x - 2xe^x + 2e^x$

$= \quad [x^2 - 2x + 2].e^x$

Self-Assessment - I

1. Solve the following :

a) $\int (x2 - 2x + 3)\ dx$

b) $\int x\ .\ \log x\ dx$

c) $\int \frac{x}{x^2 - 3x + 2} dx$

8.4 THE DEFINITE INTEGRALS

The area under a graph of a continuous function from a to b (a<b) can be expressed more succinctly as the definite integral of f(x) over the integral a to b. Mathematically.

$$\int_a^b f(x)dx = \frac{\lim}{h \to \infty} f(x)\Delta x1$$

Here the left hand side is read, "the integral from a to be of f of x dx". Here a is called the lower limit of integration and b the upper limit of integration. Unlike the indefinite integral which is a set of functions containing all the antiderivatives of f(x) as explained before, the definite integral is a real number which can be evaluated by using the fundamental theorem of calcus. This theorem staes that the numerical value of the definite integral of continuous function f(x) over the interval from a to b is given by the indefinite integral F(x) + c evaluated at the upper limit of integration b, minus the same indefinite integral F(x) + c evaluated at the lower limit of integration a. Since c is common to both, the constant of integration is eliminated in subtraction expressed mathematically

$$\int_a^b f(x)dx = F(x)\int_a^b = F(b) - F(a)$$

Where the symbol $\int_a^b$ indicates that b and a are to be substituted indicates b and a are to be substituted successively for x.

Example: the definite integral $\int_1^4 10x\,dx$ are evaluated as follows

$$\int_1^4 10x\,dx = 5x^2\Big|_4^1 \quad = 5(4)^2 - 5(1)^2$$
$$= 5x16 - 5$$
$$= 80 - 5$$
$$= 75$$

2. The definite integral $\int_1^3 (4x^3 + 6x)dx$

Evaluated

$$\int_1^3 (4x^3 + 6x)dx = [x^4 + 3x^2]_1^3$$
$$= [3^4 + 3.3^2] - [1^4 + 3.1^2]$$
$$= [81 + 27] - [1 + 3]$$
$$= 108 - 4 = 104$$

Properties of definite integrals

(i) Reversing the order of the limits changes the sign of the definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(ii) If the upper limit of integration equals the lower limit of integration the value of the definite integral is zero.

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

(iii) The definite integral can be expressed as the sum of component sub integrals.

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
$$a \le b \le c$$

(iv) The sum or difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.

$$\int_a^b f(x)dx + \int_a^b 9(x)dx = \int_a^b [f(x) \pm 9(x)]dx$$

(v) The definite integral of the constant times a function is equal to the constant times the definite integral of the function.

$$\int_0^n K(ex)dx = k\int_0^b f(x)dx$$

Numerical examples based on properties of definite integrals.

Example : Show that

$$\int_{-4}^{4}(8x^3 + 9x^2)dx = \int_{-4}^{0}(8x^3 + 9x^2)dx + \int_{0}^{4}(8x^3 + 9x^2)dx$$

$$\text{L.H.S.}\int_{-4}^{4} (8x^3 + 9x^2)dx = 8\frac{x^4}{21} + \frac{9x^3}{3}\Big|_{-4}^{4}$$

$$= 2x^4 + 3x^3\Big|_{-4}^{4}$$
$$= 2(4)^4 + 3(4)^3 - [2(-4)^4 + 3(-4)^4]$$
$$= 2(256) + (64) = [2(256) - 3(64)]$$
$$= 5129 + 192 - [512 - 192]$$
$$= 704 - 320$$
$$= 384$$

$$\text{R.H.S.} \int_{-4}^{4}(8x^3+9x^2)dx \int_{0}^{4}(8x^3+9x^2)dx$$

$$= \frac{8x^4}{4}+\frac{9x^3}{3}\Big|_{-4}^{0}+\frac{8x^4}{4}+\frac{9x^3}{3}\Big|_{-4}^{0}+\frac{8x^4}{4}+\frac{9x^3}{3}\Big|_{0}^{4}$$

$$= 2x^4+3x^3\Big|_{-4}^{0}+2x^4+3x^3\Big|_{0}^{4}$$

$$= -\left[2(-4)^4+3(-4)^3\right]+2(4)^4+3(4)^3$$

$$= -[512-192]+(512+192)$$

$$= -320+704$$

$$= 384 = \text{RHS}$$

Example : Integrate the following definite integral by means of the substitution method

$$\int_0^3 \frac{6x}{x^2+1}dx$$

Solution : Let U = $x^2+1, \frac{d4}{dx} = 2x$ and

$dx = d4/2x$ substitutive

$$\int \frac{6x}{x^2+1}dx = \int \frac{6x}{2u}dx = 3\int x.u^{-1}du$$

Integrating with respect to u

$$3\int u^{-1}du = 3\ \text{In}|u|$$

Substitutive $4 = x^2+1$

$$\int \frac{6x}{x^2+1}dx \quad = 3\ \text{In}|x^2+1|\ \Big|_0^3$$

$$= 3\ \text{In}|3^2+1| - 3\ \text{In}(0^2+1)$$

$$= 3\text{In}10 - 3\ \text{In}1, \text{since In I} = 0$$

$$= 3\ \text{in}10 = 6.9078$$

Example : Suppose we are asked to find the integral of the MC function.

$$MC = a + bQ$$

Where a and b > 0, over the Q = m to Q =n where m and n are constants, m being the smaller. This is written in the form.

$$\int_{Q=m}^{Q=n} (a + bQ)dQ \text{ or} \int_{m}^{n} (a + bQ)dQ$$

Q = m the lower limit and

Q = n the upper limit.

The indefinite form of the integral is first of all calculated and bracketed and the limits are written outside the bracket. The bracket is then evaluated when Q= m, the lower limit, and this value is subtracted from the value of the bracket when Q= n, e.g.

$$\begin{aligned}
\int_{m}^{n} (a + bQ)dQ &= \left[aQ + \frac{b.Q^2}{2} \right]_{n}^{m} \\
&= \left[aQ + 0.5bQ^2\right]_{n}^{m} \\
&= \left[an + 0.5bn^2\right] - \left[am + 0.5bm^2\right] \\
&= a(n - m) + 0.5b(n^2 - m^2)
\end{aligned}$$

The constant of integration need not be included when dealing with definite integrals.

This result gives us a measure of total variable costs when output increases from m to n. The definite integral is thus an area under a given curve since the area under the marginal cost equals total variable cost.

Self-Assessment - II

1. Evaluate

 a) $\int_{2}^{4} e^{3x + 2}\, dx$

 b) $\int_{0}^{2} 2^{x} 3^{-x}\, dx$

8.5 ECONOMIC APPLICATION OF THE INTEGRAL:

1. MARGINAL AND TOTAL COSTS :-

MC = d (TC)/dQ

Thus TC = $\int$ MC dQ + k

Hence if the MC function of a firm is MC = a + bQ

$$\begin{aligned} TC &= \int (a + bQ) \\ &= aQ + 0.5b^2Q^2 \end{aligned}$$

2. THE MPC AND THE CONSUMPTION FUNCTION

Given that the marginal propensity to consume is 0.8, i.e,

dc dy = 0.8

the consumption function will take the form

$$C = \int MPC\ dy = \int 0.8\,dy = 0.8y + k$$

The constant of integration k gives the level of consumption when income is zero.

3. MARGINAL AND TOTAL REVENUE

MR = d(TR)/dQ

Thus TR = $\int MR\ dQ$

If a firm's MR function is

MR = a - bQ.

Where a and b are positive constants, then

TR = $\int (a - bQ)dQ = aQ - 0.5bQ^2 + k$

k will equal zero because TR is zero when ϕ is zero

4. MEASURING CHANGES IN CAPITAL STOCK

Net investment I is the rate of change in capital stock i.e. dk/dt = I. If net investment is a function of time, it is possible to calculate the changes in capital stock over some period of time by finding the definite integral of I with respect to time.

If I = $\int_0^1 I\,dt \int_0^1 at^{\beta}dt = \left[\frac{at^{\beta+1}}{\beta+1}\right]_0^{t^*}$

$$= \frac{\alpha(t^*)^{\beta+1}}{\beta+1} - 0 = \frac{\alpha(t^*)^{\beta+1}}{\beta+1}$$

This is also area under the investment curve from t = 0 to t = t*.

5. CONSUMER'S SURPLUS

A demand relation gives the price at which any given quantity could be sold. However, in a competitive market the price does not reflect

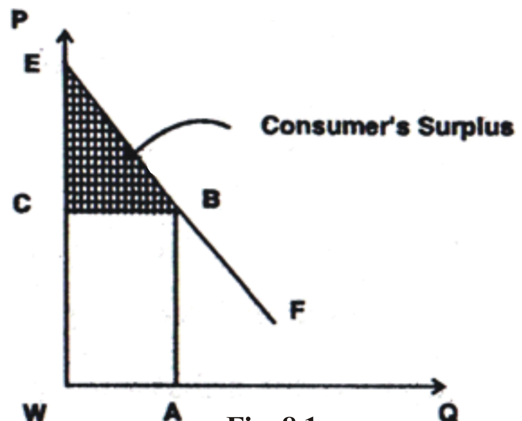

Fig. 8.1

what consumers would be willing to pay for each unit of the good rather than go without. In fact the price reflect the valuation they place on the last unit they buy.

The demand curve and Consumer's Surplus

The figure 8.1 shows a demand function EF, with price on the vertical and quantity demanded on the horizontal axis. At price WC, quantity demanded is WA. WABC is the amount paid by consumers. However the benefit by consumers is the total area under the demand function over the range WA, i.e.

WABE = benefit derived by consumer

Thus

Consumer's Surplus = WABE - WABC = CBE. Assuming the demand function is P=a - bQ, where d and b are positive constants, then

$$\text{WABE} = \int_0^{q^*} (a - bQ)dQ$$

where q* = WA

$$\int_0^{q^*} (a - bQ)dQ = \left[aQ - 0.5bQ^2\right]_0^{q^*}$$

$$= aq^* - 0.5b(q^*)^2 - a(0) - 0.5b(0)$$

$$\text{i.e.} \quad \text{WABE} = aq^* - 0.5b(q^*)2$$

$$\text{WABE} = P.Q = Pq^*$$

$$\text{But at} B, P = a - bq^*, \text{thus}$$

$$\text{WABC} = (a - bq^*)q^* = aq^* - b(q^*)^{-2}$$

$$\text{Consumer's Surplus} = aq^* - 0.5b(q^*)^2 - aq^* + bq^2$$

$$= 0.5b(q^*)^2$$

6. PRODUCERS SURPLUS

In figure 8.2 producers will supply quantity WA at WC price. At this price producer's surplus is the area of the rectangle WABC minus the area under the supply curve DE over the range WA.

Producer's Surplus = WABC - WABD = DBC

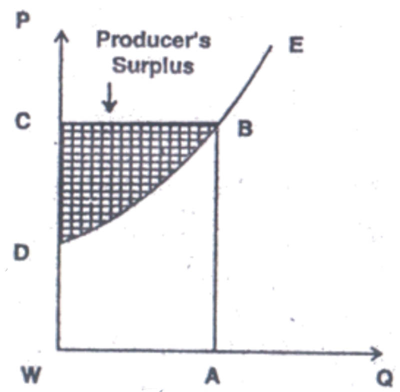

Fig. 8.2

The supply curve and producer's surplus

The supply curve is P = $(a+2Q)^2$ and WA = q*. So that

$$\int (a+2Q)^2 dQ = \frac{(a+2Q)^3}{6} + k$$

Thus

$$\text{WABD} \quad = \int_0^{q^*} (a+2Q)^2 dQ = \left[\frac{(a+2Q)^3}{6}\right]_0^{q^*}$$

$$= \frac{(a+2q^*)^3}{6} - \frac{a^3}{6}$$

WABC = P.Q = P.q* = $(a+2q^*)^2 q^*$

Producer's Surplus = $q^*(a+2q^*)^2 - \frac{(2+2q^*)^3}{6} + \frac{a^3}{6}$

$$= (a+2q^*)^2\left(\frac{4q^*-a}{6}\right) + \frac{a^3}{6}$$

The integral of $(a-2Q)^2$ with respect to Q is

$$\frac{(a-2Q)^2}{3(-2)} - k = \frac{(a-2Q)^3}{-6} + 6k$$

Applications of Integration in Economics

1. The marginal cost function of a product is given by $\frac{d\pi}{dq} = 10 - 0.01q + 0.0009q^2$

Find the total cost function and the average cost function, given that total cost = 105 when q =10.

Sol : Given $d\pi/dq = 10 - 0.01q + 0.0009q^2$

Integrating, both sides w.r.tq, we have

$$\pi = 10q - 0.01\frac{q^2}{2} + .0009.\frac{q^3}{3} + k$$

$$\pi = \log - .005q^2 + .0003.q^3 + k$$

$$\text{when}\, q = 10, \pi = 105$$

$$105 = 10(10) - .005(10)^2 + .0003(10)^3 + k$$

$$105 = 100 - .5 + .3 + k$$

$$k = 5.2$$

Hence total cost function

$$\pi = 5.2 + 10q - .005q^2 + .003q^3$$

and average cost function is

$$\frac{\pi}{q} = \frac{5.2}{q} + 10 - .005q + .0003q^2$$

2. If the marginal revenue function for output q is given by $Rm = \frac{6}{(q+2)^2} - 5$, find the total revenue function by integration. Also deduce the demand function.

Sol. We know that the total revenue function is given by

$$R = \int_0^q RM\ dq$$

The arbitrary constant k in this case is zero as the total revenue is zero at q = 0.

$$\text{Now } RM = \frac{6}{(q+2)^3} - 4$$

$$\therefore \quad R = \int_0^q \left(\frac{6}{(q+2)} - 5 \right) dq$$

$$= \int_0^q \left[\frac{6}{(q+2)^2} \right] dq - \int_0^q 5dq$$

$$= \left| \frac{(q+2)^{-1}}{-1} \right|_{q0} - 5|q|_0^q$$

$$= 6\left| -\frac{1}{q+2} \right|_0^2 - 5|q|_0^q$$

$$= 6\left\{ -\frac{1}{q+2} - 5q \right\}$$

Now, since R= pxq

$$p=\frac{R}{q}=\frac{3-\frac{6}{q+2}-5q}{q}$$

$$=\frac{3}{q}-\frac{6}{q(q+2)}-5$$

$$=\frac{3q+6-6}{q(q+2)}-5$$

$$\therefore \quad p \quad =\frac{3}{q+2}-5$$

Is the required demand function.

3. If the demand function is PD = 10-Q-Q^2

and the supply function is Ps = Q + 2

Calculate the consumer's surplus at the equilibrium price.

Sol. We have PD = 10 - Q - Q^2 Demand function

and Ps = Q +2 Supply function

At equilibrium

$$P_D = P_s$$
$$10-Q-Q^2 = Q+2 \text{ Thus}$$
$$Q^2+2Q+8=0$$
$$(Q-2)(Q+4)=0$$
$$Q=2 \text{ or} -4$$

Q cannot take a negative value, thus Q = 2

Consumer's Surplus

Area under demand curve = $\int_0^2 (10-Q-Q^2)dQ$

$$= \left| 10Q - \frac{Q^2}{2} - \frac{Q^3}{3} \right|_0^2$$

$$= \left| 20 - 2 - \frac{8}{3} = 15\frac{1}{3} \right|$$

P.Q = 8, then consumers surplus

$$15\frac{1}{3} - 8 = 7\,{}^{1}\!/_{3}$$

8.6 SUMMARY

We conclude this lesson, by summarizing what we have carried in it. Standard form integrals integral of a constant, sum, multiple, integrals by partial fraction, substitution parts definite integrals and the economic application of integrals.

8.7 LESSON END EXERCISE

Evaluate

1. $\int x^2.e^x dx$

2. $\int x\sqrt{2x+1}\, dx$

3. $\int \frac{1}{x\log x} dx$

Find the value of

a. $\int \frac{1}{\sqrt{x-1} - \sqrt{x+1}} dx$

b. $\int \frac{\log x}{x^2} dx$

c. $\int x.e^{x^2} dx$

d. $\int_1^5 x^2 dx.$

e. $\int_0^1 \log(1+x)dx$

f. $\int_1^2 \frac{x^2}{\sqrt{x-1}}dx$

4. The following are estimate of MR and MC functions

(1) MR = 4 -0.4 Q Q is output

(2) MC = 2 + 0.4 Q

Find TR when sales are (a) 4 (b) 9 in case of MR function and total variable of MR function and total variable costs when output is (a) 4 (b) 10 from MC function.

5. Given the following demand functions

(i) Q = 10 - P

Q = $64P^{-2}$

Find the consumer surplus when P = 4 & P = 5

6. The demand function of a monopolist is

3 Q = 60 - 10P

and his AC function is

$$AC = \frac{20}{Q} + 1 + 0.2Q$$

If he decides to maximize sales revenue instead of profits, show how this will effect the consumer's surplus.

8.8 SUGGESTED READINGS

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COMPARATIVE STATICS AND ALLIED ECONOMICS

STRUCTURE

9.1 INTRODUCTION

Economics models have two type of variables endogenous variables, whose value s the model is designed to explain, and exogenous variables, whose values are taken as given from outside the model. The solution values, we obtain for the endogeneous variables with typically depend on the values of the exogeneous variables, and central part of the analysis will often be to show how the solution values of the endogenous variables change with changes in the exogeneous variables. This is the problem of comparative statics equilibrium analysis or comparative statics.

9.2 OBJECTIVES :

In this lesson, we intend to take :-

- Partial differentiation and static macroeconomic models.
- A closed Macroeconomic model with government activity.
- An open Macro Model with government activity.
- Derivation of Slutsky equation.
- IS - LM Models

9.3 PARTIAL DIFFERENTIATION AND STATIC MACROECONOMIC MODELS :-

Taking a closed economic system with no government activity gives

$$Y = C+I$$

Where y is national Income, C is consumption and I is investment.

Assume I is autonomous at I* and C is a function of y, i.e C= C* +b y, where C* is the autonomous consumption and b is the marginal propensity to consume.

$$Y = C + I$$

$$Y = C^*+by + I^*$$

$$(I-b)\ y = C^* + I^*$$

$$y = \frac{c^* I}{I-b} + \frac{I^*}{I-b}$$

$$\partial y / \partial c^* \text{ or } f_{c^*} = \frac{I}{I-b}$$

f c* is the rate of change in y with respect to changes in c*; this is the multiplier

$$\partial y / \partial I^* = fI^* = I / I - b$$

Clearly f c^* and f I^* take a constant value so long as b remains constant. This is a consequence of the linear relation assumed in the model. In each case the partial derivative gives the rate of change in national income with respect to a change in any of the autonomous component party of y, assuming all other autonomous part of the consumption function increases by Δc^* then y will increase by $\Delta c^* / 1-c$. Because it analysis only the effects of the changes in autonomous variables on the equilibrium positions for the economy, and does not attempt to deal with the process of transition, this type of analysis is known as comparative statics.

9.4. A CLOSED MACROECONOMIC MODEL WITH GOVERNMENT ACTIVITY

Closed economic system with government activity, the identity now becomes

$$Y = C + I + G$$

Where G is government expenditure, I and G are assumed to be autonomous and consumption C is assumed to be a function of disposable income yd, i.e

$$C = C^* + by$$

Where C^* is the autonomous part of consumption and b is the marginal propensity to consume, yd is national income minus taxes. If

$$T = T^* + t\,y$$

Where T^* is that part of taxes, which does not depend and income and t is the tax rate, then

$$C = C^* + b\,(y-T)$$

$$C = C^* + b(y-T^* - t\,y)$$

We know,

$$y = C^* + b\,y - b\,T^* - b\,t\,y + I + G$$

$$= C^* + b\,(I - t)\,y - b\,T^* + I + G$$

$$[\,I-b\,(I-t)]\;y = C^* - b\,T^* + I + G$$

$$y = \frac{C^*}{I - b(I - t)} - \frac{bT^*}{I - b(I - t)} + \frac{I}{I - b(I - t)} + \frac{G}{I - b(I - t)}$$

$f_{T*} = - \frac{b}{I - b(I - t)}$, i.e. the rate of change in income with respect to changes in T* is constant. Consequently if T* increases by Δ T*, the national income y will decrease by

$$\Delta T^* \frac{b}{I - b(I - t)}$$

f_G gives the rate of change in y with respect to G. If G increases by Δ G then national income Y will increase by

$$\Delta G f_G = \Delta G \frac{b}{I - b(I - t)}$$

If Δ G = Δ T* i.e. an increase in expenditure is financed by an equal increase in the autonomous part of taxation, then the overall change in national income will equal

$$= \Delta T^* \frac{b}{I - b(I - t)} + \frac{I}{I - b(I - t)} \Delta G$$

$$= \Delta G \frac{(I - b)}{I - b(I - t)}$$

$$\Delta y = \frac{\Delta G(I - b)}{I - b + bt}$$

$$\text{but} \frac{(I - b)}{I - b + bt} < I$$

Since the denominator is greater than the numerator. Consequently the change in income Δy is less than the change in government expenditure ΔG, i.e.

$$\frac{\Delta y}{\Delta G} < I \text{but} > O$$

9.5 AN OPEN MACRO MODEL WITH GOVERNMENT ACTIVITY:

If the foreign sector is included in the above model the identity becomes

$$y = C + I + G + X - M$$

Where x is exports and M imports. If exports are constant, but imports M are a function of yd i.e.

$$M = M^* + m\, yd$$

Where M* is the autonomous part and m is the marginal propensity to import

$$M = M^* + m\,(y-T)$$

We know T = T*+ t y

$$\begin{aligned} M &= M^* + m(y - T^* - t\,y) \\ &= M^* + m\,y - mt\,y - T^* \\ &= M^* + m(I-t)\,y - m\,T^* \end{aligned}$$

$$y = C^* + b(I-t)\,y - b\,t^* + I + G + X - M^* - m\,(I-t)4 + m\,T^*$$

$$[I - b(I-t) + m\,(I-t)]\,y = C^* - (b-m)\,T^*\,G + I + X - M^*$$

Thus, y = $y = \dfrac{C^* - (b-m)T^* + G + I + X - M^*}{I - b(I-t) + m(I-t)}$

We have I-b (I-t)+m (I-t) = I-(I-t) (b-m)

$\therefore$ y = $y = \dfrac{C^* - (b-m)T^* + G + I + X - M^*}{I - (I-t)(C-m)}$

So f_{T^*} is constant, and

$$F_{T^*} = \frac{-(b-m)}{I - (I-t)(b-m)}$$

$$f_{C^*} = fG = ft = fx = -f_{M^*} = \frac{I}{I - (I-t)(b-m)}$$

So that f_{c^*} is also constant, if the autonomous part of imports M* increases by Δ M*, assuming all other components of y remain constant, national income will decrease by

$$\Delta M^{*} \frac{I}{I-(I-t)(b-m)}$$

If the autonomous part of the tax function T* increases by Δ T*, national income will decrease by

$$\Delta T^{*} \frac{b-m}{I-(I-t)(b-m)}$$

9.6 DERIVATION OF SLUTSKY EQUATION

Comparative statics analysis examines the effect of perturbations analysis examines the effect of perturbations on the solution variables (such as price and incomes) on the solution values for the endogenous variables (namely, quantities) changes in prices and income will normally alter the consumers expenditure pattern, but the new quantities (and prices and incomes) will always satisfy the first order conditions.

$$U = f(q_1, q_2) \text{ objective function} \qquad \text{(i)}$$

$$y = p_1 q_1 + p^2 q_2 \text{ constant equation} \qquad \text{(ii)}$$

where y is total money income, U is total satisfaction, p_1 = price of commodity one p_2 is the price of second commodity and q_1, q_2 are the amount of commodity one and two, $p_1 q_1$ is the amount of expenditure on commodity one and $p_2 q_2$ is the amount of expenditure commodity two. Here p_1, p_2 and y are exogenous.

Variables and q_1, q_2 are endogenous variables from the objective function and the constraints we form a function.

$$V = f(q_1, q_2) + \lambda (y - p_1 q_1 - p_2 q_2)$$

Which satisfy the following conditions for maximizing the total utility function.

$$\partial v / \partial q_1 = f_1 - \lambda p_1 = 0$$

$$\partial v / \partial q_2 = f_2 - \lambda p_2 = 0 \qquad \ldots\ldots\ldots \text{ (3)}$$

$$\partial v / \partial \lambda = y - p_1 q_1 - p_2 q_2 = 0$$

Now, in order to find the magnitude of the effect of price and income changes on the consumers purchases, we allow all variables to vary simultaneously. This can be done by total differentiation of the equation(3)

$$f_{11}dq_1 + f_{12}\,dq_2 = p_1\,d\lambda = \lambda dp_1$$

$$f_{21}dq_1 + f_{22}\,dq_2 = p_2\,d\lambda = \lambda dp_2 \qquad \dots\dots\dots\dots \qquad (4)$$

$$-p_1dq_1 - p_2\,d\,q_2 = -dy + q_1dp_1 + q_2dp_2$$

In order to solve this system of three equations for the three unknown (dq_1,$dq_2 d\lambda$), we take the help of Crammers rule. In matrix notation, we may write (4) as under

$$\begin{bmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{bmatrix}\begin{bmatrix} dp_1 \\ dp_2 \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1dp_1 + q_2dp_2 \end{bmatrix} \dots\dots\dots(5)$$

$$\text{or} \begin{bmatrix} dq_1 \\ dq_2 \\ d\lambda \end{bmatrix} = \frac{I}{IDI}\begin{bmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{bmatrix}\begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ -dy + q_1dp_1 + d_2dp_2 \end{bmatrix} \dots\dots(6)$$

$$\text{where}|D| = \begin{bmatrix} f_{12} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & o \end{bmatrix}$$

and D11, D12… are the co-factors of matrix [D]. By inter changing the rows and columns of the co-factor matrix, we get Adj. matrix and if we divide it by the original matrix [D] then we get the inverse of the original matrix.

Therefore,

$$dq_1 = \frac{D_{11}\lambda dp_1 + D_{21}\lambda dp_2 + D_{31}[-dy + q_1\,dp_1 + q_2dp_2]}{[D]} \quad \dots\dots\dots \quad (7)$$

$$dq_2 = \frac{D_{11}\lambda dp_1 + D_{22}\lambda dp_2 + D_{22}[-dy + q_1\,dp_1 + q_2dp_2]}{[D]} \quad \dots\dots\dots \quad (8)$$

Dividing both the sides of (7) by dp_1 and assuming that P_2 and Y do not change [or $dy = dp_2 = 0$], we have

$$\frac{\partial q_i}{op_l} = \frac{D_{11}\lambda}{[D]} + q_l \frac{D_{31}}{[D]} \qquad \ldots\ldots\ldots \qquad (9)$$

(Price effect)

The partial derivatives on the left hand side of equation (q) is the rate of change of the consumers purchase of d_I with respect to change in p_I, all other things being equal. Ceteris paribus, the rate of change with respect to income is

$$\frac{\partial q_i}{\partial y} = \frac{-D_{31}}{[D]} \qquad \ldots\ldots\ldots\ldots \qquad (10)$$

(Income effect)

Consider a price change that is compensate by an income change that leaves the consumer on his initial difference curve. An increase in the price of a commodity is, accompanied by a corresponding in his income, such that dij = 0

$$dy = \quad q_1\, dp_1 + q_2\, dp_2$$

$$-dy + q_1\, dp_1 + q_2\, dp_2 = 0$$

Hence from equation 7, we have the substitution effect

$$\left(\frac{\partial q_l}{\partial p_l}\right)_{u=\text{constant}} = \frac{D_{11}\lambda}{[D]}$$

Now, the equation (q) of slutsky equation can be written in terms of substitution effect and income effect as:

$$\left(\frac{\partial q_l}{\partial p_l}\right) = \left(\frac{\partial q_1}{\partial p_1}\right)_{u=\text{constant}} - q_l\left(\frac{\partial q_1}{\partial Y}\right)_{p=\text{constant}}$$

Price effect substitution effect income effect

9.7 IS-LM ANALYSIS

IS -LM analysis studies the equilibrium values of national income y and the interest rate r, in terms of certain policy, parameters and behavioral parameters. 'IS' stands for 'Investment Savings' and 'LM' for 'Liquidity money'.

First, consider equilibrium in the 'goods sector'. This can be represented by the equation

$$Y = C + I + G \qquad \ldots\ldots\ldots \qquad (1)$$

Which simply says that national income y is equal to the national expenditure. Expenditure can be split into consumption c, investment I, and governments spending G, the last being considered, as a policy parameter. Behavioral parameters enter when we consider how consumption is related to the national income, and how investment is affected by the interest rate. It is usual to assume that both relationships are linear, that is.

$$C = C_O + bY \qquad \ldots\ldots\ldots\ldots \qquad (2)$$

$$I = Io - a\,r \qquad \ldots\ldots\ldots\ldots \qquad (3)$$

(a, b, co, Io positive constants)

Thus the equilibrium condition becomes

$$Y = (C_O + bY) + (Io - a\,r) + G$$

$$Y - b\,Y + a\,r = Co + Io + G$$

$$(I\text{-}b)\,Y + a\,r = Co + Io + G \qquad \ldots\ldots\ldots\ldots \qquad (4)$$

Equilibrium in the money sector is represented by the statement that the supply of money is equal to the demand for money. Ms = Md. the money supply Ms is assumed to be given; it is another policy parameter. The money demand Md is assumed to depend on y and r, and again we assume a linear relationship.

$$Md = M_O + fY - g\,r \qquad \ldots\ldots\ldots\ldots \qquad (5)$$

(f, I, Mo positive constant)

Thus, the equilibrium condition becomes

$$Ms = M_O + fY - g\,r$$

$$fY - g\,r = Ms - Mo \qquad \ldots\ldots\ldots\ldots\ldots \qquad (6)$$

The two equilibrium conditions can be written as a system of two linear equation in the unknown Y and r

$$fY - g\,r = Ms - Mo \qquad \ldots\ldots\ldots\ldots \qquad (6)$$

$$(I\text{-}b)\,Y + a\,r = Co + Io + G \qquad \ldots\ldots\ldots\ldots \qquad (4)$$

In matrix terms this system is

$$\begin{pmatrix} f & -g \\ I-b & a \end{pmatrix}\begin{pmatrix} y \\ r \end{pmatrix} = \begin{pmatrix} M_s - M_o \\ C_o + g_o + G \end{pmatrix} \qquad \ldots\ldots\ldots\ldots \qquad (7)$$

and the solution is

$$\begin{pmatrix} y \\ r \end{pmatrix} = \begin{pmatrix} f & -g \\ I-b & a \end{pmatrix}^{-I} \begin{pmatrix} M_s - M_o \\ C_o + g_o + G \end{pmatrix}$$

$$= \frac{I}{fa + g(I-b)} \begin{pmatrix} a & -g \\ I-b & f \end{pmatrix}\begin{pmatrix} M_s - M_o \\ C_o + g_o + G \end{pmatrix} \qquad \ldots\ldots\ldots\ldots \qquad (8)$$

Thus, we have explicit formula for the equilibrium values y and r

$$y^* = \frac{a(M_s - M_o) + g(c_o + I_o + G)}{fa + g(I-b)}$$

$$r^* = \frac{(I-b)(M_s - M_o) + f(c_o + I_o + G)}{fa + g(I-b)}$$

These expression enable us to answer questions about what happens when the policy parameters or the behavioural parameters change.

Example: Suppose that the government decides to allow an increase in the money supply Ms. What will be the effect on the equilibrium value of the national income?

From the formula for y*, we find

$$\frac{\partial y^*}{\partial M_s} = \frac{a}{f_a + g(I-b)}$$

Thus an increase in money supply will result in an increase in y* if

$$\frac{a}{fa+g(I-b)}$$

is positive. The behavioural parameters a, b, f &g are assumed to be positive, this condition will certainly be satisfied if I-b>o that is b<I.

In fact, this condition will certainly hold in any realistic model. We have y = C+I+G, so C<Y and C= Co+ bY so bY<c , Hence bY < Y or b < I.

Note : Partial Differentiation and static macroeconomic models, IS-LM Analysis totally dependent on Anthony Martin & Norman Biggs; Mathematics for economics and finance-Methods and Modeling

Black, J & J.F. Bradley : Essential Mathematics for Economists (John Willey & Sons) Slutsky Equation, Heavily Depended on H.S. Aggarwal Book - A Mathematical Approach to Economic Theory]

9.8 SUMMARY

This chapter gives idea about the comparative statics and the allied economics by explaining the sections - 9.3, 9.4, 9.5, 9.6 and 9.7.

9.9 LESSON END EXERCISE

Q1. Explain IS-LM Model Mathematically

Q2. Examine Static Macro-Economic Modes with the help of Partial Differentiation.

Q3. Derive the Slutsky Equation.

9.10 SUGGESTED READINGS

Aggarwal, C.S. & R.C.Joshi : Mathematics for Students of Economics (New Academic Publishing Co.).

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COBB-DOUGLAS AND CES PRODUCTION FUNCTION

STRUCTURE

10.1 INTRODUCTION

Economists have developed a number of production functions. Their production functions are based on the empirical analysis of industry and agriculture in a given economy or a region. It is difficult to study all such production functions here and nor they are relevant for us. In the following analysis we discuss only the important production functions, namely (i) the Cobb-Douglas Production Function and (ii) the Constant Elasticity of Substitution Production Function or simply the C.E.S.

Production Function.

10.2 OBJECTIVES

In this lesson we shall examine Cobb-Douglas production function alongwith its properties and also examines CES production function alongwith its limitation and properties.

10.3 THE GENERAL FORM OF THE COBB-DOUGLAS PRODUCTION FUNCTION

$$X = b_0 .L^{b1}.K^{b2}.u \qquad \text{---------------} \quad (1)$$

Where x = output L = Labour input, K = capital input, u = random disturbance term, b_0 = a constant (may be treated as efficiency parameter, b^1 & b^2 positive parameters. Also L>0, K>0, b_1>0, b_2=0.

The sum of exponents (b_1+b_2) represent the degree of homogeneity (or returns to scale). In this production function the output (x) is a function of two inputs L and K or symbolically.

$$x = f(K,L) \text{ such that}$$

$$f(\lambda L, \lambda K) = (\lambda L)^b{}_1 \; (\lambda K)^b{}_2$$

$$= \lambda b_1 + b_2 \; L^{b1} \; K^{b2}$$

$$= \lambda \; b_1 + b_2 \; x.$$

Thus, if b_1+b_2= 1, the firm would be operating under the constant returns to scale and the production function is homogenous of degree one.

If $b_1 + b_2 > 1$, it will have diminishing returns to scale.

If $b_1 + b_2 > 1$, it will have increasing returns to scale.

Although the Cobb-Douglas production function is equation (i) is non-linear, it can be transformed into a linear function by converting all variable into logarithm. It is why, this function is known as a log linear function. Thus, taking the log of equation (i) on both sides, we have

$$\log x = \log b_0 + b_1 \log L + b_2 \log K + \log^u \qquad \text{--------------(2)}$$

10.3.1 Properties of Cobb-Douglas Production Function

(i) Constant returns to scale prevails in the economy. The Cobb-Douglas production function assumes that $b_1+b_2 = 1$, which means constant return to scale prevails in the economy. In other words, it proves the validity of Euler's theorem. Euler's theorem states that if the factors of production are paid according to their marginal product then total product will just exhaust. In other words, if factors are rewarded according to their marginal products, the combined share of the factors is equal to total output (x).

Since this condition is satisfied by production functions of degree one ; the Cobb-Douglas production function which was used to attempt an empirical verification of the marginal productivity. Theory of distribution of the marginal productivity theorem of distribution, statisfies Eulers theorem.

We have,

$$\text{Log } x = \text{Log } b_0 + b_1 \text{LogL} + b_2 \text{LogK} + \text{LogU}$$

$$\frac{1}{x}.\frac{\partial x}{\partial L} = \frac{b_1}{L}$$

$$\frac{\partial x}{\partial L} = x.\frac{b_1}{L}$$

$$MP_L = \frac{b_1}{L}.x$$

$$\therefore TP_L = MP_L.L = \frac{b_1}{L}.X.L. = b_1 x$$

Similarly $\frac{1}{x}.\frac{\partial x}{\partial K} = \frac{b_2}{K}$

$$.\frac{\partial x}{\partial K} = x.\frac{b_2}{K}$$

$$MP_K = \frac{b_2}{k}.x$$

$$TP_K = MP_{K.}K = \frac{b_2}{K}.X.K = b_2X$$

$$TP_{L+K} = b_1x + b_2x = (b_1 + b_2)x$$

If $(b_1+b_2) = 1$, then $TP_{L+K} = x$

(b_1+b_2) is the degree of homogeneity of the Cobb-Douglas production function. Suppose that labour and capital increased by 10 percent, then the Cobb-Douglas function becomes.

$$x = b_0(1.10L)^{b1}.(1.10K)^{b2}U$$

$$= b_0(1.10)^{b1+b2}.L^{b1}K^{b2}U$$

$$= b_0(1.10)^{b1+b2}.L^{b1}K^{b2}U$$

Thus, output would increase by $(1.10)^{b1+b2}$ and if $(b_1+b_2) < 1$ output would decrease by 10 percent, if $(b_1+b_2) > 1$, output would increase by 10 percent and if $b_1+b_2 = 1$, output would increase exactly by 10 percent. In Cobb-Douglas production function returns to scale are, therefore.

characterized by the following.

$b_1 + b_2 < 1$ = Diseconomies of scale

$b_1 + b_2 = 1$ Constant returns to scale

$b_1+b_2 > 1$ = Economies of scale

2) Elasticity of substitution is equal to one : The elasticity of substitution of the Cobb-Douglas production function is equal to unity everywhere and this is only function which satisfies this property if the production function is linear and homogenous then the elasticity of substitution $\sigma = 1$, everywhere if and only if the function is

$X = b_0.\ L^{b1}.\ K^{b2}U$ where $b_1+b_2 = 1$

or $b_2 = 1 - b_1$

Proof : We know that the elasticity of substitution can be defined as

$$\sigma = \frac{\text{\% change in factor quantity ratio}}{\text{\% change in factor price ratio}}$$

Since the rate of technical substitution between two factors is defined by (RTS)

$$= \frac{\partial K}{\partial L} = \frac{MP_L}{MP_K} = \frac{P_L}{P_K} = R$$

Then, elasticity of substitution between two factors is defined by

$$\sigma = \frac{\partial Log(K/L)}{\partial Log R}$$

$$= \frac{\partial (K/L) / \frac{K}{L}}{\frac{\partial R}{R}}$$

Where K/L = factor quantity ratio and R=factor price ratio $\frac{P_K}{P_L}$, Now we have

$$R = \frac{MP_L}{MP_K} = \frac{\partial x / \partial L}{\partial x / \partial K}$$

From the function we take the partial derivatives of x with respect to L and K respectively as :

$$x = b_0 .\ L^{b1} .\ K^{b2}\ U$$

$$\frac{\partial x}{\partial L} = b_1 . b_0 . L^{b1-1} . K^{b2} U$$

$$\frac{\partial x}{\partial K} = b_2 . b_0 . L^{b1} . K^{b2-1} . U$$

$$R = \frac{\partial x / \partial L}{\partial x / \partial K}$$

$$= \frac{b_1 b_0 . L^{b1-1} . K^{b2} U}{b_2 b_0 . L^{b1} . K^{b2-1} . U}$$

$$= \frac{b_1}{b_2}\left(\frac{K}{L}\right)$$

$$\partial R = \frac{b_1}{b_2} . \partial\left(\frac{K}{L}\right)$$

$$\frac{\partial R}{R} = \frac{\frac{b_1}{b_2} . \partial\left(\frac{K}{L}\right)}{\frac{b_1}{b_2} . \left(\frac{K}{L}\right)} = \frac{\partial (K/L)}{K/L}$$

So that $\sigma = \frac{\partial (K/L)}{K/L} / \frac{\partial (K/L)}{K/L}$

3) Expansion path generated by the Cobb-Douglas production function is linear and passes through the origin.

Proof : By first order condition for the constrained optimization, we have

$$\frac{f_L}{f_K} = \frac{P_L}{P_K}$$

We $\frac{f_L}{f_K}$ = ratio of marginal productivities and $\frac{P_L}{P_K}$ = ratio of prices.

We have found, from the function that

$$\frac{f_L}{f_K} = \frac{\partial x / \partial L}{\partial x / \partial K} = \frac{b_1}{b_2} . \frac{K}{L}$$

$$\frac{f_L}{f_K} = \frac{P_L}{P_K} = \frac{b_1}{b_2} . \frac{K}{L}$$

$$b_2 P_{L.L} = b_1 \ P_{K.K}$$

$$b_2\ P_{L.}L -_{b1}\ P_{K.}K = 0$$

This expression represents the expansion path implicit in the Cobb-Douglas production function $X = b_0.\ L^{b1}.\ K^{b2}.u$, which describes a straight line passing through the origin in the isoquant plane. The above expression can be re-written as.

$$\frac{P_L.L}{P_K.K} = \frac{b_1}{b_2}$$

The left hand side of above equation represents the share of income accruing to labour relative to that going to capital. Now that relative income shares are equal to b_1/b_2 which is determined by the technology that governs the Cobb-Douglas production function. If b_1 is high relative to b_2 then the labour share will be high relative to capital share in income. If technology is constant, if follows that a proportionate change in factor price produces a compensating proportionate change in relative factor inputs and, therefore, relative share remain constant.

4) α and β represents the labour share and capital share of the output respectively.

$$x = b_0.L^{b1}.K^{b2}.u$$

Taking the log on both the sides

$\text{Log } x = \log_{b0} + b_1\log L + b_2\log K + \log u$

$$\frac{\partial(\text{Log}x)}{\partial(\text{Log}L)} = b_1$$

$$b_1 = \frac{\partial x}{\partial L}.\frac{L}{X}$$

$$b_1 = MP_L.\ \frac{L}{X} \text{ where } MP_L = \frac{\partial X}{\partial L}$$

In perfect competition, we have

$VMP_L = P_L = P_X.\ MP_L$ where P_X = price of output

or $MP_L = P_L/P_X$

putting the value of MP_L in the above equation

We get $b_1 = \frac{P_L}{P_x} . \frac{L}{X} = \frac{\text{wage share}}{\text{Total income}}$

Similarly,

$$\frac{\partial(\log x)}{\partial(\log k)} = b_2$$

$$b_2 \frac{\partial X}{\partial K} = \frac{K}{X}$$

$$= MP_K . \frac{K}{X}$$

In perfect competition, we have

$VMP_K = P_K = MP_K.P_X$

$MP_K = P_K/P_X$

Putting the value of MP_K is the above equation,

$b_2 = P_K/P_X$. K/X = Rent Share / Total Income

Hence, b_1 = Wage share of total income.

b_2 = rent share of total income.

5) b_1 and b_2 are also the elasticities of output with respect to labour and capital.

In Cobb-Douglas production function, b1 is defined as the partial elasticity of production (x) with respect to labour (L), it denotes the percentage change in labour input, keeping capital input constant. Similarly, b_2 is defined as the partial elasticity of production (x) with respect to capital input (K), keeping labour constant.

Since b_1 and b_2 represent individually the percentage change in output given percentage change in labour and capital respectively the two co-efficients taken together measure the total change in output for a given percentage change in labour and capital.

The Cobb-Douglas production function takes a linear form, when expressed in logarithmic.

$\log x = \log b_0 + b_1 \log L + b_2 \log K + \log U$

$$\frac{\partial(\log x)}{\partial(\log x)} = b_1$$

$$b_1 = \frac{\text{(change in log of output)}}{\text{change in log of labour input}}$$

$$= \frac{\text{\% change in output)}}{\text{\% change in labour input}} \qquad \text{[capital input constant]}$$

$$= \frac{\Delta x / x}{\Delta L / L} = \frac{\Delta x}{\Delta L} \cdot \frac{L}{X} = e_{XL} \quad \text{elasticity of output w.r.t. labour}$$

Similarly $b_2 = e_{XK}$ elasticity of output w.r.t. capital of the

6) If one input is zero, output will also be zero.

The Cobb-Douglas production function assumes constant return to scale under which all the inputs are changed in equal proportion are also zero and the consequent output will also be zero.

Self-Assessment - I

1. Explain the property - I of CDPF.

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2. Elucidate the property - V of CDPF.

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10.4 THE C.E.S. PRODUCTION FUNCTION

A production function, which assumes that the basic measure of the degree of substitution is called but it not restricted 'a priori' to any value. It is called the 'Constant Elasticity of Substitution (C.E.S.) production function'. Clearly, the Cobb-Douglas and Leontief's production functions are special cases of the CES relation.

When the elasticity of substitution specified as constant, it only assumes that changes in relative factor inputs and prices do not alter the elasticity. The value of the elasticity is determined by the underlying technology ; and changes in the underlying technology affect variations on the elasticity for every level of factor inputs and prices. Hence, the constancy of the elasticity refers to its invariance with respect to changes in relative factor supplies and not to transformations of the underlying technology.

The characteristics of an abstract technology are identified by the C.E.S. production function. That is to say that it permits us to measure changes in the efficiency of a technology, changes in the technologically determined returns to scale, changes in the capital intensity of a technology and changes in the substitution of labour for capital etc.

The C.E.S. production function was derived independently by two different groups of economists: one consisting of K.J. Arrow, H.B. Chenery, B.S. Minhas and Solow; and the other group consist of Murrary Brown and De Cani. The two derivations are dissimilar; also the latter permits any degree of returns to scale. Murray Brown and De Cani (1963) used the function in an ambitious attempt to separate the effect of output change; economics of scale, technical change and changes in the relative factor prices on labour demand; with U.S. economy data over the period 1890-1958.

Further, some of the recent studies use the constant elasticity of substitution (C.E.S.) function with substitution elasticity below unity is claimed to be more suitable for production functions than the Cobb-Douglas form. Under the Cobb-Douglas production function, elasticity of substitution is equal to unity but in C.E.S. production function, as the name suggests, the elasticity of substitution is constant and not necessarily equal to unity. Four economists, Arrow, Chenery, Minhas and Solow have proposed this C.E.S. production function.

The equation of this function is.

$$X = \gamma[KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{V}{\alpha}} \qquad \ldots(1)$$

Where, X = output,

C = capital input,

N = labour input,

Υ = Coefficient of technical efficiency or scale effect (or simply the efficiency parameter). This parameter plays the same role in C.E.S. function as the coefficient A in the Cobb-Douglas function. It serves as an indicator of the general state of technology. Hence the greater the value of Υ, the larger will be the output, regardless of inputs.

K= Capital intensity factor coefficient. It is also known as the distribution parameter and is similar to in the Cobb-Douglas production function. This parameter shows the relative contribution of capital input (K) and labour input (L), to the total output (X).

(1-K) = Labour intensity coefficient.

α = It is known as the substitution parameter and is closely related to the constant elasticity of substitution. The value of e^s (elasticity of substitution) depends on the value of this parameter.

ν = It represents the degree of homogeneity of the function or the degree of returns to scale.

Hence, this function consists of three variable (X,C and N) and four parameters (Υ, K, α and ν). The variables are measured in index number terms with a common base period. And the four important parameters in the function represent the four different characteristics of an abstract technology. Thus is, in brief, the beauty of this particulars production function over the Cobb-Douglas (or other types) production function.

Relation (1) can be written in any one of the following forms.

i) $$X=\gamma[KC^{-\alpha}+(1-K)N^{-\alpha}]^{-\frac{v}{\alpha}};\alpha=-\left[1-\frac{1}{\sigma}\right]$$

ii) $$X=\gamma[KC^{\alpha}+(1-K)N^{\alpha}]^{-\frac{v}{\alpha}};\alpha'=\left[1-\frac{1}{\sigma}\right]$$

iii) $$X=\gamma\left[\frac{N^{\alpha}C^{\alpha}}{KN^{\alpha}+(1-K)C^{\alpha}}\right]^{-\frac{v}{\alpha}};\alpha=-\left[1-\frac{1}{\sigma}\right]$$

iv) $$X=\gamma[K_1C^{-\alpha}+K_2N^{-\alpha}]^{-\frac{v}{\alpha}};\alpha'=\left[1-\frac{1}{\sigma}\right]$$

where σ = elasticity of substitution

All the above four forms of the function will give us the same type of result.

10.4.1 Properties of C.E.S. Production Function:

1) The value of the elasticity of substitution depends upon the value of α We define the elasticity of substitution as

$$\sigma=\frac{\partial\left(\log\frac{K}{L}\right)}{\partial\log R}=\frac{\partial\left(\frac{K}{L}\right)}{K/L}/\frac{\partial R}{R}$$

Where K/L = factor quantity ratio, R = factor price ratio

Now our function is,

$$X=\gamma[KC^{-\alpha}+(1-K)N^{-\alpha})^{-\frac{v}{\alpha}} \quad \text{..... (2)}$$

Taking the partial derivatives of X w.r.t. N we get

$$\frac{\partial X}{\partial N}=\gamma\left(-\frac{v}{\alpha}\right)[KC^{-\alpha}+(1-K)N^{-\alpha}]^{-\frac{v}{\alpha}-1}[-\alpha N^{-\alpha-1}(1-K)]$$

$$= \gamma\nu[KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\nu}{\alpha}-1}[-\alpha N^{-\alpha-1}(1-K)] \quad \ldots..(3)$$

Now we are again using equation n(2) as

$$X = \gamma[KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\nu}{\alpha}}$$

OR

$$\frac{X}{\gamma} = [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\nu}{\alpha}}$$

OR

$$\left[\frac{X}{\gamma}\right]^{-\frac{\alpha}{\nu}} = [KC^{-\alpha} + (1-K)N^{-\alpha}]$$

OR

$$\left[\frac{X}{\gamma}\right]^{-\frac{\alpha}{\nu}\left(-\frac{\nu}{\alpha}-1\right)} = [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\nu}{\alpha}-1}$$

OR

$$= \left[\frac{X}{\gamma}\right]^{1+\frac{1}{\nu}} = [KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\nu}{\alpha}-1} \quad \ldots.. (4)$$

Putting the value of (4) in equation (3) we get,

$$\frac{\partial X}{\partial N} = \gamma\nu\left(\frac{X}{\gamma}\right)^{1+\frac{\alpha}{\nu}} N^{-\alpha-1}(1-K)] = \gamma^{-\frac{\alpha}{\nu}}\nu X^{1+\frac{\alpha}{\nu}} N-\alpha-1(1-K) \ldots.. (5)$$

Similarly we can find,

$$\frac{\partial X}{\partial C} = \gamma^{\frac{\alpha}{\nu}}\nu X^{1+\frac{\alpha}{\nu}} C^{-\alpha-1}(K) \quad \ldots\ldots (6)$$

Now we can define the marginal rate of substitution as

$$\text{MRS} = \frac{\partial X}{\partial C} / \frac{\partial X}{\partial N} = R$$ [Because MRS is equal to the price ratio].

Putting the values from equations (5) and (6) we have,

$$\therefore \quad R = \frac{\gamma^{-\frac{\alpha}{\nu}} \nu X^{1+\frac{\alpha}{\nu}} C^{-\alpha-1} K}{\gamma^{-\frac{\alpha}{\nu}} \nu X^{1+\frac{\alpha}{\nu}} N^{-\alpha-1} (1-K)}$$

$$\therefore \quad \text{Denoting} \frac{K}{1-K} = K' \text{ and} \frac{N}{C} = U \text{ we get, } R = K' U^{1-\alpha}$$

Taking the log on both the sides,

log R = log K'+(1+ α) log U

OR $\frac{1}{R}\frac{\partial R}{\partial U} = (1+\alpha)\frac{1}{U}$ or $\frac{\partial R}{R} = (1+\alpha)\frac{dU}{U}$

OR $\frac{\partial U}{U} / \frac{\partial R}{R} = \frac{1}{(1+\alpha)}$ where $\alpha > 0$. or $\sigma = \frac{1}{1+\alpha}$

Also $\alpha = \left[\frac{1}{\sigma} - 1\right] = \left[1 - \frac{1}{\sigma}\right]$ ...... (7)

If $\alpha = 0$ then $\sigma = 1$ and we shall get the situation of Cobb-Douglas production function. Hence the value of σ depends upon the value of α. It assumes whatever the value of σ may be, (depending upon α) but it is always constant.

2) The marginal products should be positive:

Our function is, $X = \gamma[KC^{-\alpha} + (1-K)N^{-\alpha}]^{-\frac{\nu}{\varepsilon}}$

Or Now $MP_L = \frac{\partial X}{\partial N} = \gamma^{-\frac{\alpha}{\nu}} \nu X^{1+\frac{\alpha}{\nu}} N^{-\alpha-1} (1-K)$

[Please see property No.1 for its calculation)

Or $$\frac{\partial X}{\partial N} = h_1 X^{1+\frac{\alpha}{v}} N^{-\alpha-1} \text{ where } h_1 = \gamma^{-\frac{\alpha}{v}} v(1-K) \qquad .....(8)$$

If there are constant returns to scale then $v = 1$ (Here v refers to returns to scale or the degree of homogeneity of the function).

If $$v = 1 \text{ then}, \frac{\partial X}{\partial N} = h_1 X^{1+\alpha} N^{-\alpha-1}$$

$$= h_1 \left(\frac{X}{N}\right)^{1+\alpha} = h_1 \left(\frac{X}{N}\right)^{\frac{1}{\sigma}} \text{ where } \sigma = \frac{1}{1+\alpha} \qquad .....(9)$$

Similarly, the Marginal Product of Capital is,

$$MP_K = \frac{\partial X}{\partial C} = \gamma^{-\frac{\alpha}{\gamma}} vX^{1+\frac{\alpha}{v}} C^{-\alpha-1}.K \qquad \text{[See property No. 1]}$$

$$= h_1 X^{1+\frac{\alpha}{v}} C^{-\alpha-1} \text{ where } h_2 = Kv\gamma^{-\frac{\alpha}{v}}$$

Again, if there are constant returns to scale, then v=1 and

$$\frac{\partial X}{\partial C} = h_2 X^{1+\alpha} C^{-\alpha-1}. \qquad \text{or } \frac{\partial X}{\partial C} = h_2 \left(\frac{X}{C}\right)^{\frac{1}{\sigma}}$$

Where $\sigma = \frac{1}{1-\alpha}$ [See property No. 1] ...(10)

It is now obvious that the marginal products are positive.

3) The marginal product curves are sloping downward or

$$\frac{\partial^2 X}{\partial N^2} < 0 \text{ and } \frac{\partial^2 X}{\partial C^2} < 0$$

4) The marginal product of each factor will increase for increase in the other factor inputs.

To satisfy this property, the RTS (or MRS) or labour (N) for capital © can be derived in terms of C.E.S. production function by taking the ratio of MP_L and MP_K as :

$$\frac{\partial X}{\partial C} / \frac{\partial X}{\partial N} = R = K' U^{1-\alpha} \qquad \text{where} \qquad K' = \frac{K}{1-K} U = \frac{N}{C}$$

[See property No.1]

Or $$R = K' U^{1/\alpha} \quad \text{where} \qquad \sigma = \frac{1}{1+\alpha} \qquad ....(11)$$

This plays an important role in the following analysis. If the production process is highly L-intensive (small capital is used per unit of labour), the MP_L is high compared to MP_K for each N/C ratio ; thus a unit reduction in the labour rate has to be compensated for larger increase in the rate of capital than if the process were to be less L-intensive. In this sense, K is a measure of capital intensity.

If σ is high, then capital is easily substitute for labour and vice-versa. The expression for R tells us that if we reduce the rate capital inputs by one unit, we have to increase the rate of labour input by more when factors are easily substitutable for each other then when they are ceteris paribus. Perhaps the easiest way to rationalize this is to recall that the more easily substitutable are the factors for each other, the more similar they are forms for each other, the more similar they are form an economic point of view. If σ is low then the factors are dissimilar.

10.4.2 Advantages of C.E.S. Production Function over the Cobb-Douglas Production Function and Limitations :

Advantages of CES Production Function

1) C.E.S. function represents a production function where all types of returns may be analyze 1. Since is not necessarily be equal to one ($\sigma \neq 1$), it represents

a more general form of production techniques.

2) As we have seen, the C.E.S. production function takes into consideration a number of important parameters. It, therefore, covers a wide range of variety, substitutability, and efficiency.

3) Estimation of this C.E.S. function is very easy. Of course, some transformation is needed, if we write output per unit of labour as a function of capital per unit of labour, that is,

$$\frac{X}{L} = \int\left(\frac{K}{L}\right)$$

then this production function becomes very easy.

4) It removes all the difficulties of Cobb-Douglas production function and free from the unrealistic assumptions of the Cobb-Douglas production function.

Limitations of the C.E.S. Production Functon :

1) The C.E.S. production function combines in one parameter, ν, two forces that affect it. In the first place, economies of scale can result from an expansion in the scale of operation for a given technology. Alternatively, given the scale of operation, a technological change can alter the rate of output. In empirical application both forces may affect the homogenity parameters, ν, and it may not be possible to distinguish between them.

2) Prof. H. Uzawa has attempt this function and concludes that it is difficult to generalize it to n-factors of production.

3) A limitation of the C.E.S. production function is associated with its principal virtue - the specification of elasticity of substitution which is invariant to changes in factor proportions. Recall that we have allowed the elasticity of substitution (σ) to changes in response to changes in factor proportions. But this is an 'a priori specification': we really do not know whether the elasticity of substitution (σ) should vary when factor proportions change. It is true structure prescribes a variable elasticity due to changes in factor proportions and we claim that

elasticity in changing for technological reasons then we are ascribing to technological change more than is due to it. Unless a completely general function is specified - a polynomial of degree n-it seems this difficulty must be accepted. Since it is impossible with the available data and statistical techniques to obtain estimates of completely general production and since they do not necessarily satisfy all neo-classical criteria (properties of C.E.S. function), we are forces for the immediate future to utilize the C.E.S. production function and live with the potential specification error.

4) A fourth difficulty to this C.E.S. function is that K, the capital intensity parameter is not dimensionless.

Beside from there theoretical difficulties, there is an empirical problem; the C.E.S. production function is relatively difficult to fit to data.

[Note: Both the Cobb-Douglas and CES production function heavily depended on H.S. Agarwal book - A Mathematical approach to Economic Theory]

Self-Assessment - II

1. Explain the advantages and disadventages of C.E.S.P.F.

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2. Narrate the first property of C.E.S.P.F.

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10.5 SUMMARY

This chapter deals with the non-Linear Production Functions and narrate

their importance to the reader.

10.6 LESSON END EXERCISE

Q1. Explain the Cobb-Douglas production function and its properties.

Q2. Elasticity of substitution is equal to one in case of Cobb-Douglas production function.

Q3. Examine C.E.S production function and its properties.

10.7 SUGGESTED READINGS

Aggarwal,C.S & R.C. Joshi : Mathematics for Students of Economics (New Academic Publishing Co.).

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Kandoi B : Mathematics for Business and Economics with Applications (Himalaya Publishing House).

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